# Embedding 3-manifolds in 4-space and link concordance via double branched covers

by

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# Abstract

The double branched cover is a construction which provides a link between problems in knot theory and other questions in low-dimensional topology. Given a knot in a 3-manifold, the double branched cover gives a natural way of associating a 3-manifold to the knot. Similarly, the double branched cover of a properly embedded surface in a 4-manifold is a 4-manifold whose boundary is the double branched cover of the boundary link of the surface. Consequently, whenever a link in $S^3$ bounds certain types of surfaces, its double branched cover will bound a 4-manifold of an appropriate type.

The most familiar situation in which this connection is used is the application to slice knots as the double branched cover of a smoothly slice knot is the boundary of a smooth rational ball. Examples of 3-manifolds which bound rational balls can therefore easily be constructed by taking the double branched covers of slice knots while obstructions to a 3-manifold bounding a rational ball can be interpreted as slicing obstructions. This thesis is primarily concerned with two different extensions of this idea.

Given a closed, orientable 3-manifold, it is natural to ask whether it admits a smooth embedding in the four-sphere $S^4$. Examples can be obtained by taking the double branched covers of doubly slice links. These are links which are cross-sections of an unknotted embedding of a two-sphere in $S^4$. Certain links can be shown to be doubly slice via ribbon diagrams with appropriate properties. Other embeddings can be obtained via Kirby calculus.

On the other hand, many obstructions to a 3-manifold bounding a rational ball can be adapted to give stronger obstructions to embedding smoothly in $S^4$. Using an obstruction based on Donaldson's theorem on the intersection forms of definite 4-manifolds, we determine precisely which connected sums of lens spaces smoothly embed. This method also gives strong constraints on the Seifert invariants of Seifert manifolds which embed when either the base orbifold is non-orientable or the first Betti number is odd. Other applicable

methods, also based on obstructions to bounding a rational ball, include the $d$ invariant from Ozsváth and Szabó's Heegaard-Floer homology and the Neumann-Siebenmann $\overline{\mu}$ invariant. These are used, in conjunction with some embedding results derived from doubly slice links, to examine the question of when the double branched cover of a 3 or 4 strand pretzel link embeds.

The fact that the double branched cover of a slice knot bounds a rational ball has a second interpretation in terms of knot concordance. In this viewpoint, the double branched cover gives a homomorphism from the concordance group of knots to the rational cobordism group of rational homology 3-spheres. This can be extended to a concordance group of links using a notion of concordance based on Euler characteristic. This yields link concordance groups which contain the knot concordance group as a direct summand with an infinitely generated complement. The double branched cover homomorphism extends to large subgroups containing the knot concordance group.

# Acknowledgements

Firstly, I'd like to thank Brendan Owens, whose support and advice has been extremely valuable to me in the past few years. I have enjoyed my time in the department, in no small part due to the many fine people who are, or have been, here. I would also like to acknowledge the support of my family and thank my brother, Gordon, for occasionally listening to me talk about topology.

# Declaration

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 4 is based on joint work with B. Owens [DO12].

# Contents

# Chapter 1

# Introduction

Let $Y$ be a 3-dimensional, closed manifold. A *knot* $K$ in $Y$ is a smooth isotopy class of a smooth embedding of a circle $S^1 \hookrightarrow Y$. A *link*, with $m$ components, is an isotopy class of an embedding of a disjoint union of $m$ circles in $Y$. Classical knot theory is the study of knots and links in $S^3$. Important questions in knot theory concern properly embedded surfaces in 4-manifolds. A knot in $S^3$ is called (smoothly) *slice* if it is the boundary of a (smoothly) properly embedded disk $D^2$ in $D^4$.

An important invariant of a knot or link $K$ is the double branched cover, $\Sigma_2(Y, K)$. This is the 3-manifold which is a double branched cover of $Y$ with branch set $K$. It can be constructed by taking a double cover of the knot complement $Y \setminus \nu K$ and gluing in a solid torus along the boundary so that on a meridian of $K$, we have the double covering map $S^1 \to S^1$ given by $z \mapsto z^2$. See for example [KT76]. This construction extends in two obvious ways. Firstly, we can take an $n$-fold cyclic covering by replacing 2 with $n$. Secondly, and for this thesis more importantly, we can take the (double) branched cover of a properly embedded surface in a 4-manifold. This means that the double branched cover gives a connection between questions about knots and links in $S^3$ and closed, orientable 3-manifolds. Figure 1.1 illustrates the idea – a relationship between a knot and a surface it bounds is reflected in the 3 and 4-manifolds obtained by taking branched covers.

Perhaps the most striking example of the usefulness of this procedure is the application to the issue of which knots are smoothly slice. The key fact is that the double branched cover of $D^4$ over a properly embedded disk is a rational homology ball – a 4-manifold $U$ with $H_*(U; \mathbb{Q}) = H_*(D^4; \mathbb{Q})$ [CG86]. We can therefore show that a knot is not slice by showing that the double branched cover cannot bound such a manifold.

An obstruction to a 3-manifold bounding a rational ball can be obtained from Donald-

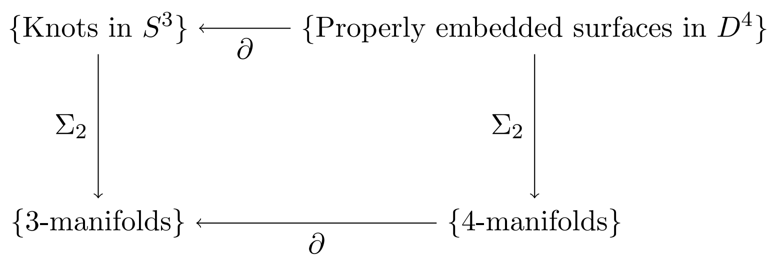



Figure 1.1: **A relationship between knots, surfaces and low-dimensional manifolds.**

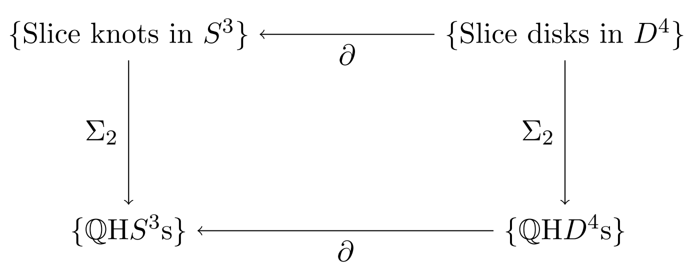



Figure 1.2: **Diagram for slice knots.**

son's theorem on the intersection forms of smooth definite 4-manifolds [Don87]. This has been used to produce obstructions to knot sliceness (for example [CG88, Lis07a, Lis07b, GJ11, Lec12, Wil08]). Lisca has used this method, in conjunction with explicit examples of smooth slice disks, to determine precisely when a 2-bridge knot is slice. Work of Greene-Jabuka and Lecuona gives similar results for certain families of 3-strand Montesinos knots.

## 1.1 Examples of links and double branched covers

In this section, we describe some families of knots and 3-manifolds which will be of importance in this thesis. We fix notation and specify relationships between these families.

### 1.1.1 Two-bridge links and lens spaces

A link $L$ is called a *two-bridge link* if it has a projection in $\mathbb{R}^2$ with two maxima and minima. All two-bridge links are determined by a rational number $\frac{p}{q} \geq 1$, where $\gcd(p, q) = 1$. This is a knot if $p$ is odd and has two components if $p$ is even.

A diagram of the link $S(p, q)$ can be drawn as follows. We find a *negative continued fraction* of $\frac{p}{q}$. This is a set of integers $a_1, \dots, a_n$ such that

$$\frac{p}{q} = [a_1, \dots, a_n]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

and draw the diagram shown in Figure 1.3, where a box marked $a_i$ indicates that the two strands are twisted so that there are $a_i$ crossings. These crossings are positive or negative according to the sign of $a_i$. If we take a different continued fraction for $\frac{p}{q}$, the resulting link is isotopic. (See, for example, [BZ03, Chapter 12] [Sch56].) Figure 1.4 shows $S(-23, 7)$.

The double branched covers of two-bridge links are lens spaces.

**Definition 1.1.** *Let $p$ and $q$ be coprime integers. The lens space $L(p, q)$ is the 3-manifold resulting from Dehn surgery on the unknot in $S^3$ with slope $-\frac{p}{q}$.*

There is a diffeomorphism $L(p, q) \cong L(p, q + np)$ for each $n \in \mathbb{Z}$. This means that we can assume that $p > q > 0$, unless $|p| \leq 1$ or $q = 0$. Changing the orientation gives $-L(p, q) \cong L(p, p - q)$. Taking $p = \pm 1$ or $q = 0$ gives $S^3$ and $p = 0$ gives $S^1 \times S^2$. It is often convenient to exclude these from the class of lens spaces.

The double branched cover of the two-bridge link $S(p, q)$ is $L(p, q)$. We can describe both the links and the 3-manifolds in terms of the *plumbing* construction, which we briefly

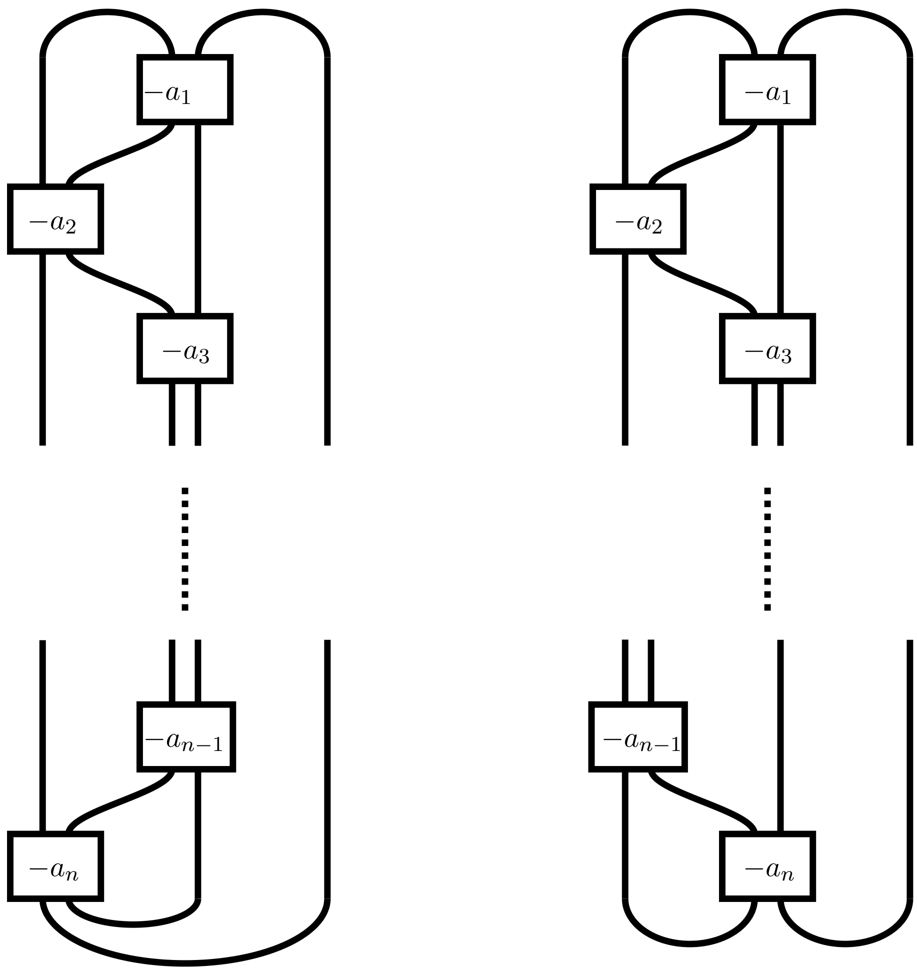



Figure 1.3: **Two-bridge link corresponding to $[a_1, \ldots, a_n]^-$; on left for even $n$ and on right for odd $n$.**

summarise. Let $X_1$, $X_2$ be 2-disk bundles over closed surfaces. These can be plumbed together by choosing disks $D_1$ and $D_2$ in the base and identifying the bundles over these disks – $D_1 \times D^2$ and $D_2 \times D^2$ – via a map exchanging the factors. The resulting 4-manifold is said to be obtained by plumbing $X_1$ and $X_2$. See [GS99, Example 4.6.2] for details on plumbings. A 4-manifold can be described by a weighted tree by taking a disk-bundle over $S^2$ for each vertex with Euler number given by the weight and performing plumbing on a pair of these bundles whenever the corresponding vertices are connected by an edge. The boundary of the 4-manifold obtained by plumbing on a linear tree with weights $-a_1, \ldots, -a_n$ is $L(p, q)$ where $\frac{p}{q} = [a_1, \ldots, a_n]^-$.

A lower-dimensional analogue replaces $D^2$-bundles over surfaces with isotopy classes of embedded $D^1$-bundles over $S^1$ in $S^3$. Figure 1.5 shows a plumbing of two untwisted bands. If these bands are plumbed according to a linear weighted tree, the boundary of the resulting surface in $S^3$ is the two-bridge link $S(p, q)$, where $\frac{p}{q}$ is given as a negative continued fraction by the weights.

Let $X$ be a plumbed 4-manifold defined by a weighted tree. This is a 2-handlebody[1] and so it is simply connected. The two-handles correspond to vertices in the graph. If we take these as a basis for $H_2(X)$, a matrix for the intersection form of $X$ is given by the incidence matrix of the graph, where the diagonal entries are given by the weights and the off-diagonal entries are zero or one depending on whether or not there is an edge connecting the vertices.

For a lens space $Y = L(p, q)$, with either orientation, there is a negative definite plumbing $X$ with $\partial X = Y$. We can always find a continued fraction $\frac{p}{q} = [a_1, \ldots, a_n]^-$ with each $a_i \geq 2$ and the plumbing along a linear graph with weights $-a_i$ gives $X$. We will refer to this as the *standard (negative) definite plumbing* corresponding to $Y$. Figure 1.6 shows such a diagram for a manifold whose boundary is a connected sum of lens spaces.

### 1.1.2 Montesinos links, pretzel links and Seifert manifolds

A *Montesinos link* is one obtained by plumbing bands according to a star-shaped weighted graph, while a *Seifert manifold* over $S^2$ is the boundary of a 4-manifold produced by plumbing disk bundles over $S^2$ according to a star-shaped graph.

Let $\Gamma$ be a star-shaped graph with central vertex $v$ and $n$ legs. If $v$ is weighted by $r$ and the $i^{th}$ leg has weights $a_1^i, \ldots, a_{m_i}^i$ where $a_1^i$ is adjacent to the centre and $a_{m_i}^i$ is a leaf

[1] We use the term '2-handlebody' to refer to a 4-manifold produced by attaching 2-handles to $D^4$.

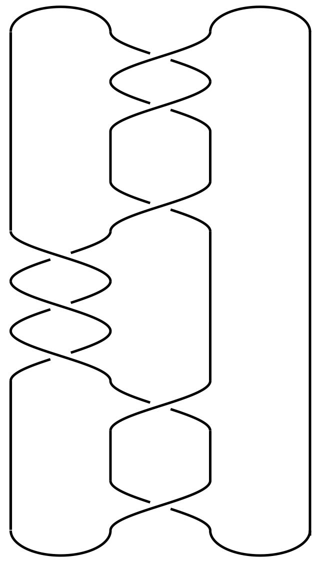


Figure 1.4: **Two-bridge knot** $S(-23, 7)$**.**

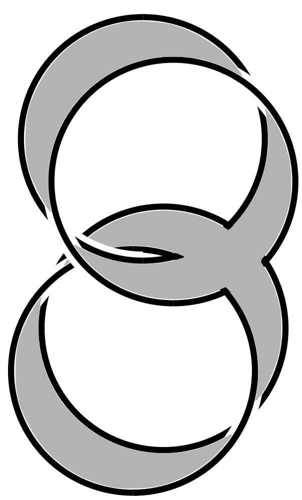


Figure 1.5: **Plumbing of two (untwisted) bands.**

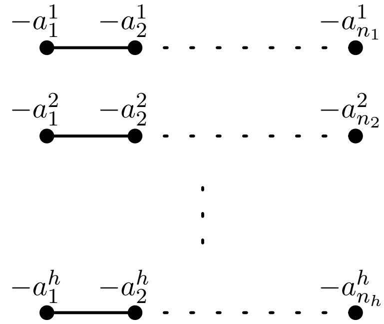



Figure 1.6: **Plumbing graph for a negative definite 4-manifold with boundary a connected sum of lens spaces.**

of Γ, we let $\frac{a_i}{b_i} = [a_1^i, \ldots, a_{m_i}^i]^-$. The Montesinos link produced from this diagram will be denoted by

$$M(r; (a_1, b_1), \ldots, (a_n, b_n)).$$

The *Seifert manifold* or *Seifert fibred space* obtained from this plumbing will be denoted by

$$Y(S^2; r; (a_1, b_1), \ldots, (a_n, b_n)).$$

As was the case for 2-bridge links, the double branched cover of a Montesinos link is a Seifert manifold [OwSt06, Mon73]. Linear trees are a special case of star-shaped ones, so 2-bridge links are also Montesinos links and lens spaces are Seifert manifolds. A surgery diagram is shown in Figure 1.7. We will call the surgery curve with coefficient $r$ the 'central' curve. The set $S = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ is called the set of Seifert invariants of the Seifert manifold.

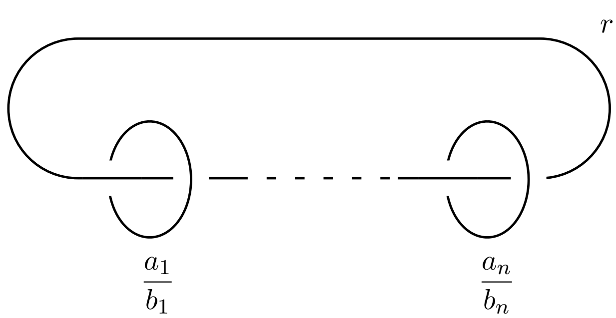



Figure 1.7: $Y(S^2; r; (a_1, b_1), \ldots, (a_n, b_n))$.

Let $F$ be a closed, connected but possibly non-orientable surface. A *Seifert manifold* with base surface $F$ is one given by the boundary of a plumbing along a star-shaped graph where the central vertex corresponds to a disk-bundle over $F$ and every other vertex corresponds to disk-bundle over a sphere. This manifold is denoted

$$Y(F; r; (a_1, b_1), \ldots, (a_n, b_n)).$$

where $r$ is the central framing and the Seifert invariants $(a_i, b_i)$ again come from the legs of the graph. The surface $F$ is called the base surface or base orbifold.

Surgery diagrams for these Seifert manifolds can obtained by modifying diagrams for Seifert manifolds with the same Seifert invariants and base surface $S^2$. The base surface can be modified by adding $T^2$ or $\mathbb{RP}^2$ summands. The surgery diagram changes at the central curve as shown in Figures 1.8 and 1.9.

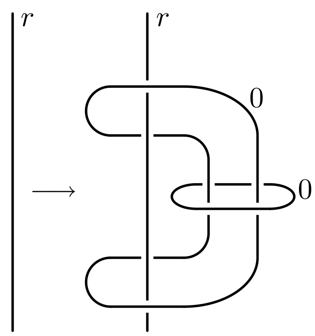



Figure 1.8: **Adding a $T^2$ summand.**

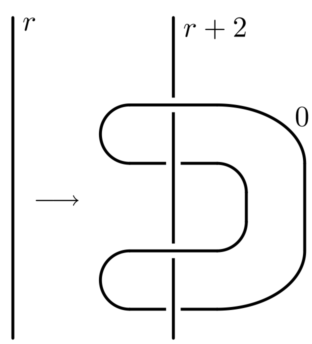



Figure 1.9: **One way of adding an $\mathbb{RP}^2$ summand.**

Another local picture for adding an non-orientable summand can be obtained from Figure 1.9 by reversing the crossings and changing the central framing to $r-2$, see [CH98, Appendix]. A more symmetrical picture is shown in Figure 1.10.

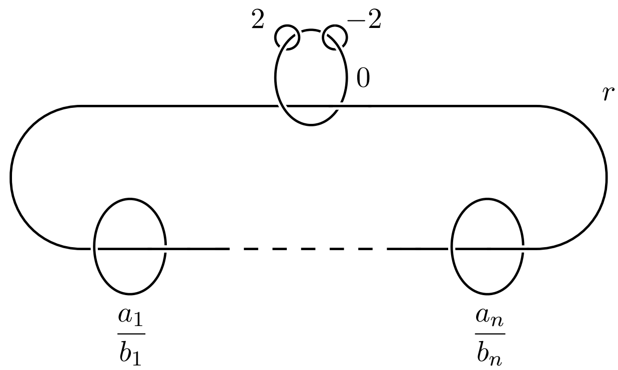



Figure 1.10: $Y(\mathbb{RP}^2; r; (a_1, b_1), \ldots, (a_n, b_n))$.

**Definition 1.2.** *Let* $Y = Y(F; r; (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ *be a Seifert manifold with base* $F$*. The generalised Euler invariant is*

$$e(Y) = \sum_{i=1}^{n} \frac{b_i}{a_i} - r.$$

Note that there is a diffeomorphism

$$Y(F; r; (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)) \cong Y(F; r + 1; (a_1, a_1 + b_1), (a_2, b_2), \ldots, (a_n, b_n)),$$

given by a Rolfsen twist. This preserves the generalised Euler invariant. We may use this diffeomorphism to choose a preferred notation for a given Seifert manifold. It is sometimes adjusted so that $r = 0$ [NR78], [CH98] but we can also arrange that each $a_i > 1$.

When $Y$ is a Seifert manifold with base orbifold $S^2$ and $e(Y) > 0$ a negative definite 4-manifold with boundary $Y$ can be obtained by a standard plumbing construction [NR78]. The same construction gives a semi-definite 4-manifold when $e = 0$. Figure 1.11(c) shows this plumbing for $Y(S^2; 0; (3, 1), (3, -1), (3, 1))$.

With a minor modification, we can find negative definite 4-manifolds whose boundaries are Seifert manifolds with any base orbifold. We describe these 4-manifolds in Proposition 3.10. In particular, for a non-orientable base surface, we obtain a negative definite 4-manifold regardless of $e(Y)$.

We now define some terms to describe some special types of Seifert invariants.

**Definition 1.3.** *Two Seifert invariants are* equivalent *if they are of the form* $(a, b)$ *and* $(a, b + na)$ *for some integer* $n$*. A* complementary pair *of Seifert invariants are a pair equivalent to* $(a, b)$ *and* $(a, -b)$*. A* weakly complementary pair *is a pair which is either* *a complementary pair or equivalent to* $(a,b),(a,-b')$ *where* $bb' \equiv 1 \bmod a$. *A Seifert invariant* $(a,b)$ *is called* odd *if* $a$ *is odd and* even *if* $a$ *is even.*

A special case of Montesinos links are *pretzel links*. These are the Montesinos links which arise from a plumbing tree where the central vertex has weight zero and every leg has length one[2]. The pretzel link $P(a_1,\dots,a_n)$ consists of $n$ strands with $a_i$ twists each, joined in a chain at the bottom and the top. The link in Figure 1.11 is $P(3,-3,3)$.

The double branched covers of pretzel links are Seifert manifolds with base $S^2$ and Seifert invariants of the form $(a_i,b_i)$ with $b_i = \pm 1$ and $r = 0$. In this case, the legs in the standard negative definite plumbings will have a simpler form. Every leg will either consist of single vertex with a negative weight or a chain of vertices, all with weight $-2$. We will denote these manifolds as $Y(a_1b_1,\dots,a_nb_n)$. We will also assume $n \geq 3$ as this gives a lens space when $n \leq 2$. Integer surgery diagrams are shown in Figure 1.14 when $n = 3,4$. The manifold $Y(a_1,\dots,a_n)$ is the double branched cover of the pretzel link $P(a_1,\dots,a_n)$.

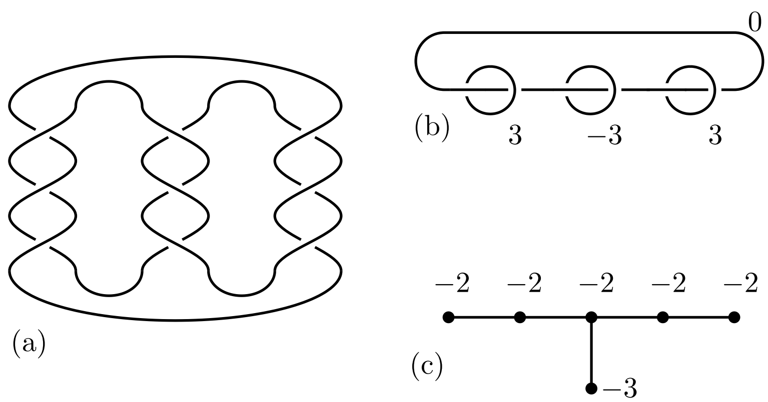



Figure 1.11: **(a) The pretzel knot $P(3,-3,3)$; (b) a surgery diagram for its double branched cover $Y(3,-3,3)$; (c) the plumbing graph for a negative definite 4-manifold with boundary $Y(3,-3,3)$.**

Figure 1.12 summarises the relationships between these families of links and manifolds.

Let $S$ be a set of Seifert invariants. These define a connected sum of lens spaces $L = Y(S^2;\infty;S)$. There is a cobordism between $L$ and $Y(S^2;r;S)$ for each $r$ given by adding a central curve.

[2]The first condition here is somewhat unnecessary as if each leg has length one, we can arrange that the central weight is zero by adding new legs with weights $\pm 1$.

$$
\begin{array}{ccccc}
\{\text{2-bridge links}\} & \subset & \{\text{Montesinos links}\} & \supset & \{\text{Pretzel links}\} \\
\Big\downarrow{\scriptstyle \Sigma_2} & & \Big\downarrow{\scriptstyle \Sigma_2} & & \\
\{\text{Lens spaces}\} & \subset & \{\text{Seifert manifolds with } F = S^2\} & \subset & \{\text{Seifert manifolds}\}
\end{array}
$$

Figure 1.12: **Relationships between families of links and manifolds.**

**Definition 1.4.** *Let $S = \{(a_i, b_i)\}_{i=1}^{h}$ be a set of Seifert invariants. Define $W_{S,r}$ to be the 2-handle cobordism between $L = -\#_{i=1}^{h} L(a_i, b_i)$ and $Y = Y(S^2; r; S)$.*

## 1.2 Embedding 3-manifolds in $S^4$

One of the main problems considered in this thesis is that of when a closed (and necessarily orientable) 3-manifold embeds smoothly in $S^4$. By a result of Wall [Wal65] (see also [Hir61] for the case of orientable 3-manifolds), every closed 3-manifold embeds in $S^5$. For $S^4$, results are known for special classes of manifolds including some Seifert fibred cases [GL83], [CH98], some of which also hold for topological locally flat embeddings. In the case of smooth embeddings, the question was examined systematically in [BB12]. A key observation in our approach to this problem is that the double branched cover of a *smoothly doubly slice* link $L$ – one which is a cross-section of a smooth unknotted 2-sphere in $S^4$ – embeds smoothly. In Lemma 2.4, we will see that there is a diagram as shown in Figure 1.13.

$$
\begin{array}{ccc}
\{\text{Doubly slice links in } S^3\} & \xrightarrow[\iota]{} & \{\text{Unknotted spheres in } S^4\} \\
{\scriptstyle \Sigma_2}\Big\downarrow & & {\scriptstyle \Sigma_2}\Big\downarrow \\
\{\text{3-manifolds}\} & \xrightarrow[\iota]{} & S^4
\end{array}
$$

Figure 1.13: **Diagram for doubly slice links and embeddings in $S^4$.**

We can use this method to find examples of 3-manifolds which embed. In Chapter 2 we show certain links are doubly slice via ribbon diagrams and so we can embed manifolds by taking double branched covers. The setup is similar to that for slice knots in Figure 1.2. We use methods adapted from slice obstructions, in fact obstructions to a 3-manifold bounding a rational ball, to get obstructions to embedding in $S^4$. Notably, an obstruction based on Donaldson's diagonalisation theorem and used by Lisca and others can be adapted. In Chapter 3 we apply this and obtain the following results.

**Theorem 3.16.** *Let $L = \#_{i=1}^h L(p_i, q_i)$. Then $L$ embeds smoothly in $S^4$ if and only if each $p_i$ is odd and there exists $Y$ such that $L \cong Y \# -Y$.*

This generalises a result of Gilmer-Livingston [GL83] and Fintushel-Stern [FS87] in the case $h = 2$.

**Theorem 3.17.** *Let $Y$ be a Seifert manifold with non-orientable base surface $F$. If $Y$ embeds smoothly in $S^4$ then the Seifert invariants of $Y$ occur in weak complementary pairs. In addition, whenever there are Seifert invariants $(a_i, b_i), (a_j, b_j)$ with $a_i, a_j$ both even, then $a_i = a_j$ and $b_i \in \{\pm b_j, \pm b_j'\}$.*

While this result does not put any restriction on the Euler invariant of $Y$, it is shown in [CH98] that for a given set of Seifert invariants there are only finitely many possible values of $e(Y)$ for which an embedding is possible and, in the case of complementary pairs with every $a_i$ odd, these are completely described.

We also consider orientable base surfaces. An interesting special case, considered by Hillman [Hil09], occurs when $e(Y) = 0$. These are the only examples where $b_1(Y)$ is odd.

**Theorem 3.32.** *Let $Y$ be a Seifert manifold with orientable base surface $F$ and $e(Y) = 0$. If $Y$ embeds smoothly in $S^4$ then the Seifert invariants of $Y$ occur in complementary pairs.*

**Remark 1.5.** This holds even for topological embeddings when $F = S^2$ [Hil09].

When $Y$ has complementary pairs of Seifert invariants with every $a_i$ odd and $e(Y) = 0$ it embeds smoothly in $S^4$ [CH98].

Other methods also apply. The obstruction from Donaldson's result does not appear to give as strong constraints for other families of Seifert manifolds. In Chapter 3 we also employ obstructions using the correction term from Heegaard-Floer theory and the Neumann-Siebenmann $\overline{\mu}$ invariant of a spin structure, a lift of the Rochlin invariant.

The former, as shown in [GJ11], gives a useful strengthening of the obstruction from Donaldson's theorem.

These are used to establish a result on the double branched covers of pretzel links with three or four strands.

**Theorem 3.42.** *Let $Y$ be of the form $Y(a,b,c)$ or $Y(a,b,c,d)$ where $a,b,c \in \mathbb{Z}\backslash\{-1,0,1\}$ and $d \in \mathbb{Z}\backslash\{0\}$. If $Y$ embeds smoothly in $S^4$ then it is (possibly orientation-reversing) diffeomorphic to one of the following*

- $Y(a,-a,a)$;
- $Y(a,-a,a,-a)$;
- $Y(a,-a,b,-b)$ *with $b$ odd;*
- $Y(a \pm 1,-a,a,-a)$;
- $Y(2\lambda-1,-2\lambda-1,-2\lambda^2)$.

*In addition, all but the last of these do embed smoothly in $S^4$.*

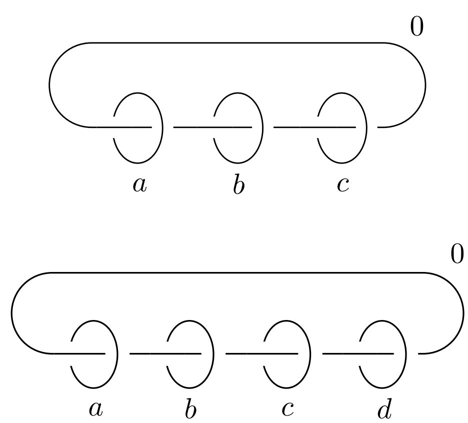



Figure 1.14: $Y(a,b,c)$ **and** $Y(a,b,c,d)$**.**

Note that $Y(a,b,\pm1)$ and $Y(a,b,\pm1,\pm1)$ are lens spaces so the constraints imposed in the above statement are merely for convenience. Some of the manifolds considered by Theorem 3.42 have $e(Y)=0$. In particular, we see that the converse of Theorem 3.32 is not true in general.

## 1.3 Concordance of links

A natural extension of the notion of slice knots is the concordance group of knots $\mathcal{C}$ [FM66]. Given a pair of oriented knots $K_1, K_2$ in $S^3$, we can form the connected sum $K_1 \# K_2$ in $S^3 \# S^3 = S^3$. In a diagram, this amounts to drawing the knots disjointly; deleting a small arc from each of the knots, which by isotopy can be assumed to be adjacent, and then connecting the start of each arc to the endpoint of the other by a pair of parallel arcs which do not introduce any new crossings to the diagram. Figure 1.15 shows a connected sum of two-bridge knots. This operation turns the set of knots in $S^3$ into a monoid. The unknot gives the identity.

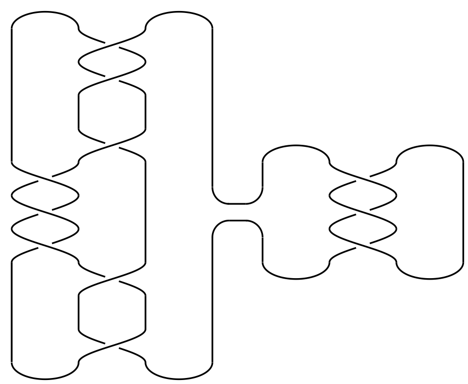


Figure 1.15: **Connected sum of** $S(-23, 7)$ **and** $S(3, 1)$**.**

An equivalence relation on the set of knots is given by $K_1 \sim K_2$ if and only if $-K_1 \# K_2$ is a (smoothly) slice knot, where $-K_1$ denotes the mirror image of $K_1$ with the opposite orientation. This is preserved by connected sum so the quotient of the monoid of knots by this relation gives a group, $\mathcal{C}$. Allowing locally flat slice disks as well as smooth ones gives a related group $\mathcal{C}_{TOP}$, but we will mainly consider the smooth version. This group is a widely-studied object in knot theory. See, for example, the survey article [Liv05] for more background.

The double branched cover gives a homomorphism $\Sigma_2$ from $\mathcal{C}$ to the smooth rational cobordism group of rational homology 3-spheres. This is the set of classes of oriented rational homology spheres under the relation $Y_1 \sim Y_2$ if $-Y_1 \# Y_2$ smoothly bounds a rational ball and the group operation is connected sum.

A sensible way to expect that $\mathcal{C}$ can be generalised is by allowing links in $S^3$ rather than just knots. The aim should be to define a concordance group of links which retains many of the properties of $\mathcal{C}$. In particular, since knots are just 1-component links, it should contain a subgroup given by $\mathcal{C}$ and the double branched cover homomorphism should, if possible, extend to the link group.

Chapter 4, which is based on joint work with B. Owens [DO12], defines link concordance groups $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ and examines their properties.

The essential ideas of these groups can be sketched as follows. In order to get a sensible notion of connected sum, we require links to have a specific oriented component along which to take the sum. However, we have a choice about whether or not to orient the entire link. Both are reasonable so we consider *partly oriented* links with just one oriented component and *marked oriented* links where every component is oriented but one is selected to define connected sums. A link $L \subset S^3$ is called $\chi$-*slice* if it is the boundary of a properly embedded surface $F$ (with no closed components) in $D^4$ with $\chi(F) = 1$.

Proposition 4.1 establishes that this is compatible with $\Sigma_2$.

$$\begin{array}{ccc}
\{\chi\text{-slice links with } \det \neq 0\} & \xleftarrow{\ \partial\ } & \{\chi\text{-slice surfaces in } D^4\} \\
\Big\downarrow \Sigma_2 & & \Big\downarrow \Sigma_2 \\
\{\mathbb{Q}\mathrm{H}S^3\mathrm{s}\} & \xleftarrow{\ \partial\ } & \{\mathbb{Q}\mathrm{H}D^4\mathrm{s}\}
\end{array}$$

Figure 1.16: **Diagram for $\chi$-slice links.**

By considering a slightly more restricted class of surfaces, we get equivalence relations, called $\chi$-concordance, on the monoids of partly oriented or marked oriented links with connected sums. The main results are as follows.

**Theorem 4.13.** *The set of $\chi$-concordance classes of partly oriented links forms an abelian group*

$$\mathcal{L} \cong \mathcal{C} \oplus \mathcal{L}_0$$

*under connected sum which contains the smooth knot concordance group $\mathcal{C}$ as a direct summand. The inclusion $\mathcal{C} \hookrightarrow \mathcal{L}$ is induced by the inclusion of oriented knots into partly oriented links.*

*The complement $\mathcal{L}_0$ of $\mathcal{C}$ in $\mathcal{L}$ contains a $\mathbb{Z}/2$ direct summand and a $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty$*

*subgroup.*

**Theorem 4.16.** *The set of $\chi$-concordance classes of marked oriented links forms an abelian group*

$$\widetilde{\mathcal{L}} \cong \mathcal{C} \oplus \widetilde{\mathcal{L}}_0$$

*under connected sum which contains the smooth knot concordance group $\mathcal{C}$ as a direct summand (with $\mathcal{C} \hookrightarrow \widetilde{\mathcal{L}}$ induced by the inclusion of oriented knots into marked oriented links). Forgetting orientations on nonmarked components induces an epimorphism $\widetilde{\mathcal{L}} \to \mathcal{L}$. We obtain group homomorphisms, which are induced from maps on the monoids of knots, partly oriented and marked oriented links.*

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \tilde{\mathcal{L}} \\ & \searrow & \downarrow \\ & & \mathcal{L} \end{array}$$

*The complement $\widetilde{\mathcal{L}}_0$ of $\mathcal{C}$ in $\widetilde{\mathcal{L}}$ contains a $\mathbb{Z} \oplus \mathbb{Z}/2$ direct summand and a $\mathbb{Z}^\infty$ subgroup.*

In Chapter 4 we define $\chi$-concordance and the groups $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. Maps from these groups using the double branched cover are considered and so are topological versions.

# Chapter 2

# Constructing embeddings of 3-manifolds in $S^4$

The main purpose of this chapter is to establish the following:

**Theorem 2.1.** *The following manifolds embed smoothly in $S^4$:*

1. $\#_{i=1}^{h}\left(L(a_i,b_i)\#L(a_i,a_i-b_i)\right)\#n(S^1\times S^2)$ *with each $a_i$ odd;*
2. $Y(F;0;(a,1),(a,-1),(a,1),\ldots,(a,\pm 1))$ *with $F$ orientable;*
3. $Y(F,r,S)$ *with $F$ orientable, $S$ a set of complementary pairs of odd Seifert invariants and $r$ such that $e(Y)=0$;*
4. $Y(F,r,S)$ *with $F$ non-orientable of genus $k$, $S$ a set of complementary pairs of odd Seifert invariants and $r$ such that $e(Y)\in\{2k,2k-4,\ldots,-2k\}$;*
5. $W_{S,0}$ *as defined in Definition 1.4 when $S$ is a set of complementary pairs of odd Seifert invariants;*
6. $Y(S^2;0;(a,1),(a,-1),(a,1),(an+1,-n))$*;*
7. $Y(S^2;0;(a,1),(a,-1),(b,1),(b,-1))$ *with $b$ odd;*
8. $Y(S^2;0;(4,1),(4,1),(12,-7))$.

**Remark 2.2.** Some of these families contain special cases which are interesting in their own right, such as simple 3-manifolds like $T^3$ and $S^1\times S^2$. Double branched covers of pretzel links appear in cases 2, 3, 6 and 7. We also note that there is some overlap between some of these families.

**Remark 2.3.** Some of these embeddings are known already, by different methods to the ones used here. Crisp-Hillman [CH98] construct embeddings in cases 2, 3 and 4. The embeddings in case 1 follow from Zeeman [Zee65]. Embeddings of some manifolds in family 6 are described in [BB12] as 'deform-spun' embeddings.

The proof of Theorem 2.1 occupies the rest of the section. We will construct embeddings for these cases separately.

## 2.1 Constructing embeddings via doubly slice links

This section will describe how to use doubly slice links to produce smooth embeddings of 3-manifolds in $S^4$.

An embedding of $S^n$ in $S^{n+2}$ is unknotted if it is the boundary of an embedded $D^{n+1}$. We will call a link $L$ in $S^3$ (smoothly) doubly slice if it is a cross-section of an unknotted (smooth) embedding of $S^2$ in $S^4$.

**Lemma 2.4.** *Let $L$ be a link in $S^3$ and $Y$ be the $n$-fold cyclic branched cover of $S^3$ with branch set $L$. If $L$ is smoothly doubly slice then $Y$ smoothly embeds in $S^4$.*

*Proof.* The $n$-fold cyclic branched cover of $S^4$ with branch set an unknotted $S^2$ is $S^4$. This comes from repeated suspension of the unbranched $n$-fold cover of $S^1$ over itself, where the branched covering map is extended in the obvious way (see [Rol76, Example 10.B.4]).

If $L$ is doubly slice then the pair $(S^3, L)$ sits inside $(S^4, S^2)$. The preimage of this subset gives $Y$ embedded in $S^4$. □

A source of doubly slice knots is Zeeman's twist-spinning construction [Zee65]:

**Theorem 2.5.** *Let $K$ be any knot. Then $K\# -K$ is doubly slice.*

The special case when $K$ is a 2-bridge knot is of particular interest. The double branched cover of a 2-bridge knot is a lens space $L(p, q)$ with $p$ odd. All such lens spaces arise in this way so applying Zeeman's result to connected sums of 2-bridge knots gives the embeddings in Theorem 2.1 (1).

To produce more examples of doubly slice links we look at embeddings of spheres into $S^4$.

Let $f : S^2 \to S^4$ be a smooth embedding of a sphere $S$. We may delete a point in $S^4$ away from $S$. Then let $r : \mathbb{R}^4 \to \mathbb{R}$ be a projection such that $r \circ f$ is a Morse function for

$S$. The preimage of each $t \in \mathbb{R}$ describes a link in $\mathbb{R}^3$, which will be denoted $S_t$, except at the isolated critical values of $r \circ f$. The isotopy type of these links only change when we pass through one of these critical values. At a minimum or maximum of the Morse function the link changes by the addition or removal of an unknotted component while at a saddle point the cross-section changes by a band move.

We will use the following theorem of Scharlemann:

**Theorem 2.6** (Main theorem of [Sch85])**.** *Let $\gamma_1$ and $\gamma_2$ be knots such that some band move on the split link $L = \gamma_1 \cup \gamma_2$ gives the unknot. Then $\gamma_1$ and $\gamma_2$ are unknots and the band move is the connected sum.*

From this, the following result can be obtained.[1]

**Proposition 2.7.** *Let $S$ be a sphere in $S^4$. Suppose there is a projection $r$ so that the level sets of $S$ are such that $S_0$ is an unknot; all of the maxima occur at some level $t > 0$; all of the minima occur at levels $t < 0$ and every cross-section is a completely split link.*

*Then $S$ is an unknotted sphere.*

Note that, by Scharlemann's result, all of the level sets are unlinks and at every saddle point the number of components increases as $|t|$ increases.

*Proof.* The proof is by induction on the number of saddle points, $n$. The case $n = 1$ follows easily from Scharlemann's result – we may assume the sphere has two minima and one maximum and so the band move is just the connected sum of a pair of unknots. This describes an unknotted sphere.

Suppose $S$ has $n$ saddle points. It can be arranged that they occur at distinct levels. Let $t_n$ be the level of the top one. In order to increase the number of components, the band move at $t_n$ will just affect one of the components, $K$. By an isotopy, it can be arranged that the maxima capping off all the other components of the unlink here occur at level $t' < t_n$.

Choose some $t$ such that $t' < t < t_n$. The cross-section $S_t$ gives an unknot so there is a 2-disk $D$ at this level. Surgery along $D$ gives spheres $S'$ and $S''$. The Morse function of $S$ induces Morse functions on these spheres with 2 and $n - 1$ saddle points respectively. By induction, both are unknotted so bound 3-cells $D'$ and $D''$ respectively. These give $\overline{D} = D' \cup_D D''$, a 3-cell bounded by $S$. □

[1] A similar statement appears in [Hos68]. The proof contains a gap which is repaired by [Sch85].

**Corollary 2.8.** *Suppose $L$ is a link in $S^3$ and there are two sets of band moves $\{A_i\}_{1\leq i\leq k}$ and $\{B_j\}_{1\leq j\leq l}$ such that performing the moves*

- $\{A_i\}_{1\leq i\leq k} \cup \{B_j\}_{1\leq j\leq l}$ *gives an unknot;*
- $\{A_i\}_{1\leq i\leq k} \cup \{B_j\}_{j\leq n}$ *gives an $l-n+1$-component unlink $(0 \leq n \leq l)$ and*
- $\{A_i\}_{i\leq n} \cup \{B_j\}_{1\leq j<l}$ *gives an $k-n+1$-component unlink $(0 \leq n \leq k)$.*

*Then $L$ is doubly slice. In addition, a link obtained by performing any subset of this entire collection of band moves is doubly slice.*

*Proof.* The above proposition can be applied to show that these band moves describe an unknotted sphere. Take the unknot obtained by using all of the bands as the central level set and undo the $A$ bands in order above it to get unlinks in the level sets above. Doing the same with the $B$ bands below gives an unknotted sphere. Changing the order of the band moves simply takes a different cross-section of the same sphere so the result follows. □

A schematic diagram is shown in Figure 2.1. This shows a sphere with two saddle points. The horizontal cross-sections are unlinks and show that the sphere is unknotted but other cross-sections give doubly slice links $L$.

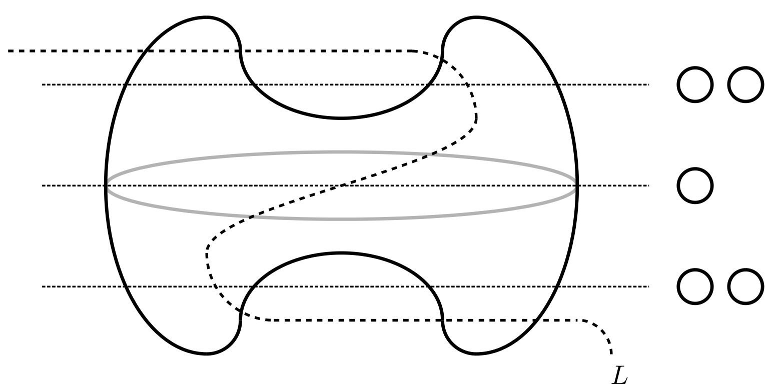



Figure 2.1: **A schematic diagram – the link $L$ is a cross-section of an unknotted sphere.**

We will use this result to produce families of doubly slice links. First, we illustrate the method with a simple example involving only two band moves.

**Example 2.9.** The pretzel knot $P(3,-3,3)$ is doubly slice.

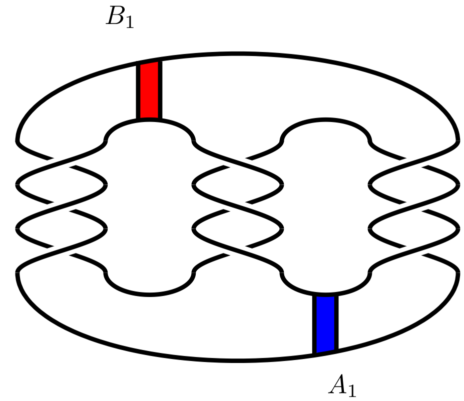



Figure 2.2: **Band moves on** $P(3, -3, 3)$**.**

Figure 2.2 shows that necessary band moves and Figure 2.3 shows the level sets of the unknotted sphere. After performing either band move, we get the pictures on the left or right of Figure 2.3 and we can 'unwind' the crossings to see that the result is an unlink. Similarly, the knot obtained after both band moves is an unknot. This shows that $P(3, -3, 3)$ is doubly slice.

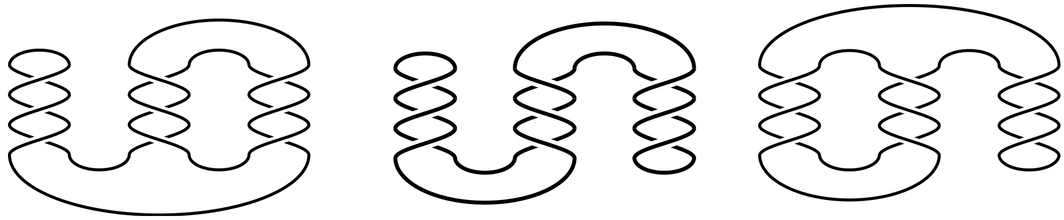


Figure 2.3: **Sequence of unlinks describing the level sets of a trivial sphere.**

This diagram resembles one in [Hos68] where this was done in the absence of a complete proof of Scharlemann's result, Note that it extends in the obvious way to $P(a, -a, a)$. A picture showing $P(2, -2, 2)$ as a cross-section of an unknotted sphere appears in [Fox61].

This example is a special case of the following. A box marked with $a$ denotes a pair of strands with $a$ half-twists.

**Proposition 2.10.** *Let* $L_{a,n}$ *be the link in Figure 2.4. It is doubly slice for any* $a, n \in \mathbb{Z}$.

*Proof.* We ignore band $C$ for the moment.

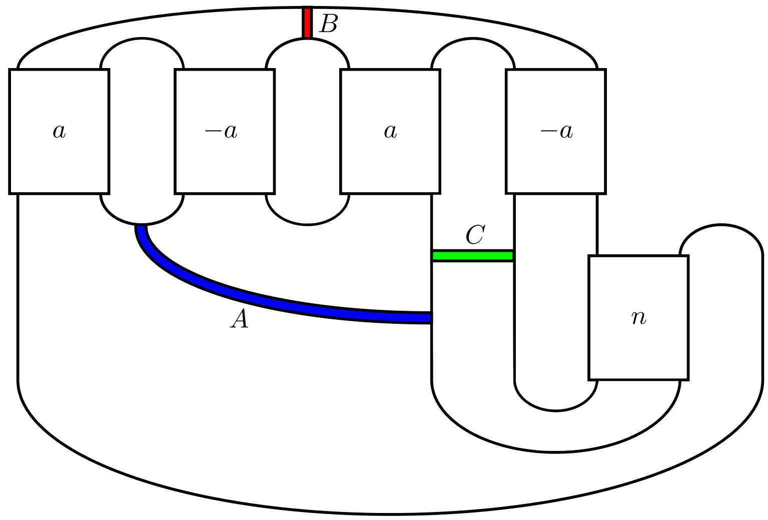



Figure 2.4: **Band moves on $L_{a,n}$.**

After performing band move $A$, the crossings in $-a$ and $a$ twists in the second and third strands can be cancelled in pairs. The first and fourth strands may then also be removed so this gives a 2-component unlink.

Band move $B$ has a similar effect and also gives a 2-component unlink. Applying both band moves gives an unknot so we may apply Corollary 2.8.

□

**Corollary 2.11.** *The pretzel links $P(a,-a,a)$, $P(a,-a,a,-a\pm 1)$ and $P(a,-a,a,-a)$ are all doubly slice for any $a \in \mathbb{Z}$.*

*Proof.* The first two of these families are of the form $L_{a,n}$ when $n = 0$ or $\pm 1$. The unknotted sphere in Proposition 2.10 can be extended using band $C$. If we do band moves $B$ and $C$ we get a 3-component unlink so the three bands describe an unknotted sphere with three saddle points. The link given by band move $C$, $P(a,-a,a,-a)$, is therefore also doubly slice. □

To construct more doubly slice links, we need to reprove Zeeman's theorem for 2-bridge knots. We begin with the following intermediate result.

**Lemma 2.12.** *Let $K$ be a $(2,2k+1)$-torus knot $T_{2,2k+1}$ for $k \geq 1$. Then $K\# -K$ is a cross-section of the unknotted sphere shown in Figure 2.5, where the $2k$ bands are labelled as in Corollary 2.8.*

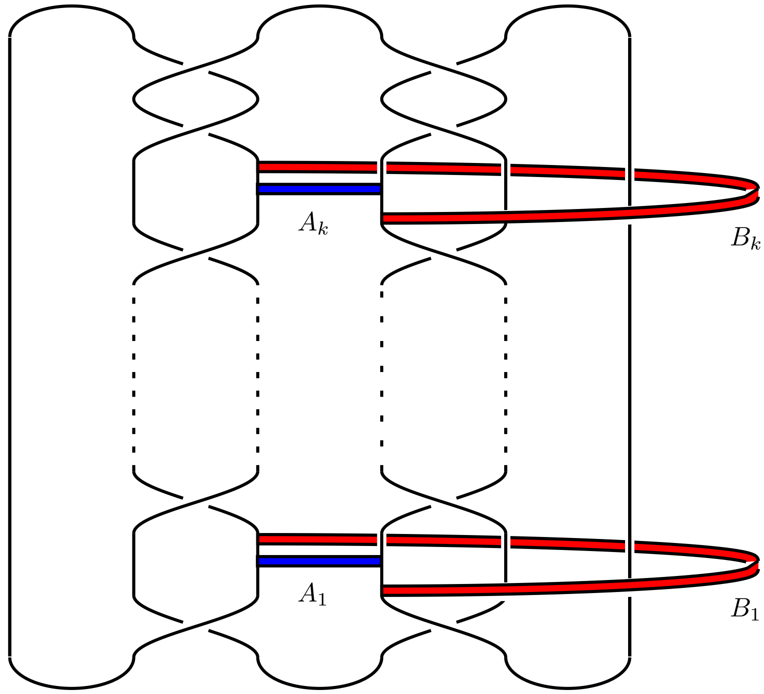



Figure 2.5: **Band moves on** $T_{2,2k+1}\# - T_{2,2k+1}$**.**

*A similar picture (with two bands) shows the same fact in the trivial case of* $T_{2,1}\#{-}T_{2,1}$.

*Proof.* We must verify that the bands in this picture satisfy the conditions of Corollary 2.8. First, we claim that performing band moves $A_1$ and $B_1$ changes the sign of the crossing immediately above the pair of bands. The effect of band $A_1$ is shown in Figure 2.6 and there is an isotopy giving Figure 2.7. Band move $B_1$ gives two pairs of canceling crossings and so transforms the knot to $T_{2,2k-1}\# - T_{2,2k-1}$. The rest of the bands are unaffected so we may continue this process with $k$ such pairs of band moves to produce the unknot.

Now suppose we do all of the $A$ band moves and $B_1, \dots B_n$ for some $n < k$. We begin by noting that when $i \leq n$ each pair $(A_i, B_i)$ cuts down the number of crossings, as before. It is therefore enough to show that applying the $k$ band moves $A_1, \dots A_k$ to the diagram for $T_{2,2k+1}\# - T_{2,2k+1}$ gives a $k+1$ component unlink.

The band move $A_1$ gives a 2-component unlink as can be seen in Figure 2.6 – all of the crossings can be cancelled in pairs. Immediately after performing each subsequent band move, a further unlinked component can be removed.

There is an isotopy of Figure 2.5 which moves each band $B_i$ into the position that $A_i$ is drawn in. This can be seen by rotating the second factor in the connected sum anti-clockwise by $2\pi$ through an axis passing through the band of the connected sum. This

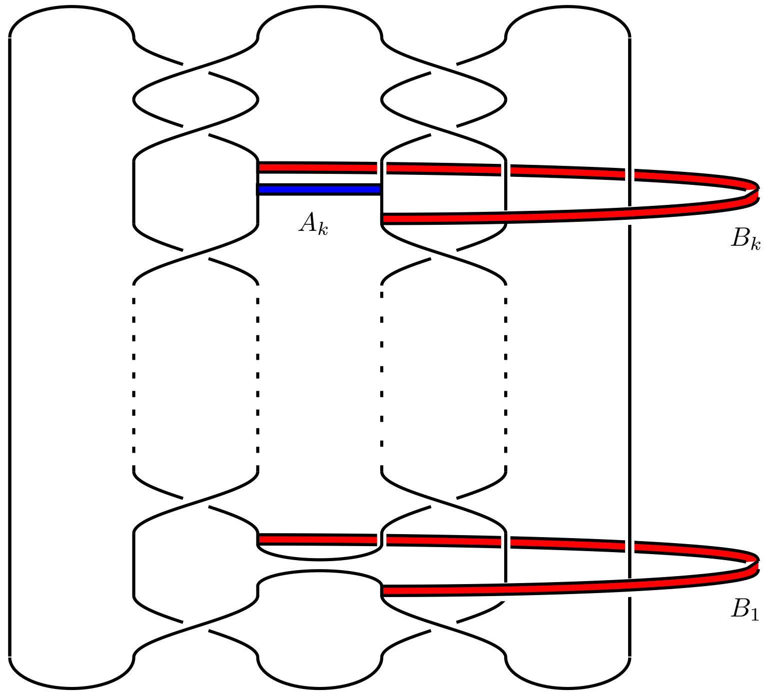



Figure 2.6: **The result of band move $A_1$.**

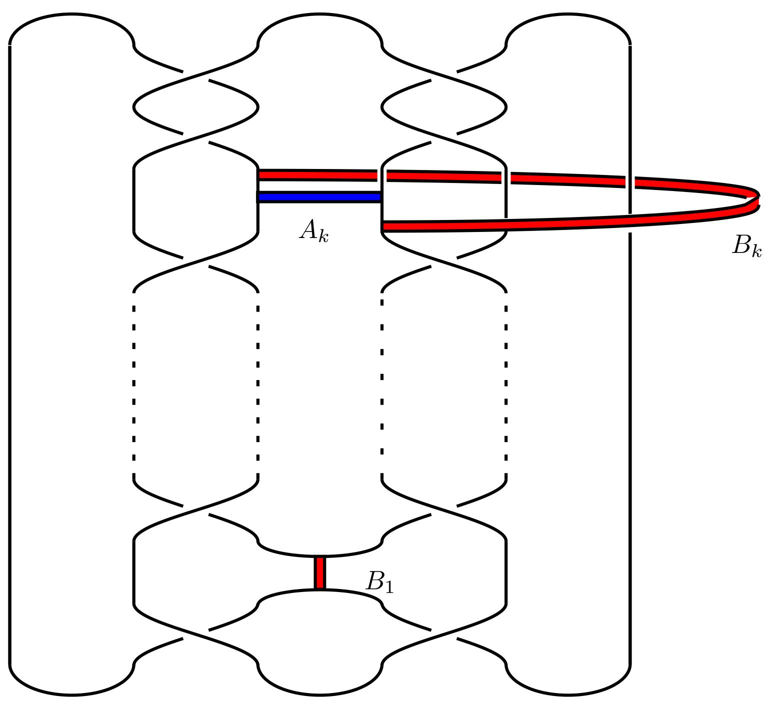



Figure 2.7: **Isotopy simplifying band $B_1$ .**

symmetry establishes that the above argument also works with each $A_i$ replaced by $B_i$, and so verifies the remaining condition in Corollary 2.8. □

We now show that $P(a, -a, b, -b)$ is doubly slice when $b$ is odd. There are two cases, which we consider separately.

**Proposition 2.13.** *The link $P(a, -a, b, -b)$ is doubly slice when $a$ is even and $b$ is odd.*

*Proof.* Figure 2.8 shows that there is a band move using a band $C$ on $P(a, -a, b, -b)$ which gives $T_{2,|b-a|}\# - T_{2,|b-a|}$. Since $b - a$ is odd, Lemma 2.12 gives band moves on this knot satisfying Corollary 2.8. We can extend this picture by adding the band move $C$ and interpreting it as $B_0$.

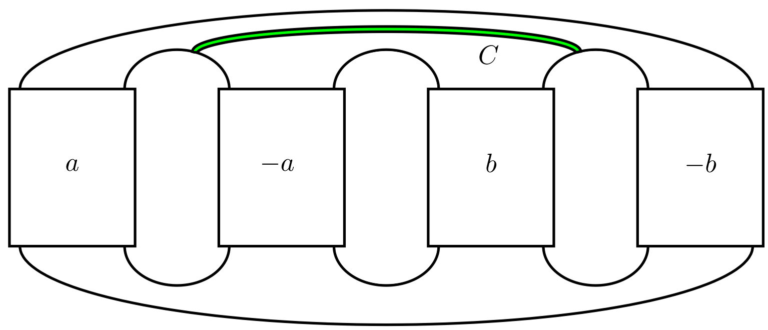



Figure 2.8: **A band move on** $P(a, -a, b, -b)$ **.**

We claim that this picture also satisfies the conditions of Corollary 2.8. All but one of the cross-sections which need to be considered are obtained by applying a set of band moves including $C$ and so are described by Lemma 2.12. Therefore the only thing that remains to be checked is that applying all of the band moves $A_i$ without $C$ gives an unlink with one more component than the one obtained by including $C$.

This is exhibited by Figure 2.9, with $2k + 1 = |b - a|$.

□

**Proposition 2.14.** *The link $P(a, -a, b, -b)$ is doubly slice when $a$ and $b$ are both odd.*

*Proof.* We proceed in the same manner in Proposition 2.13. The band $D$ in Figure 2.10 turns the link into the sum $T_{2,a}\# - T_{2,a}\#T_{2,b}\# - T_{2,b}$. We find band moves for this knot using Lemma 2.12, and the fact that a connected sum of unknotted spheres is also unknotted. Let $a = 2l + 1$ and $b = 2k + 1$. We obtain the diagram shown in Figure 2.11.

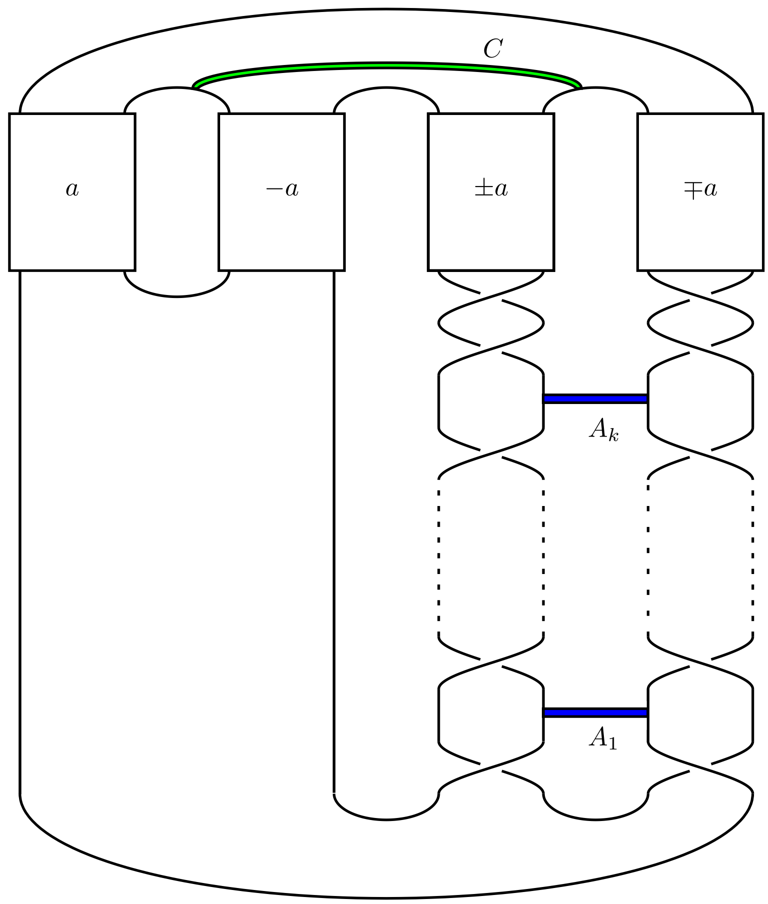



Figure 2.9: **Bands $A_i$ and $C$.**

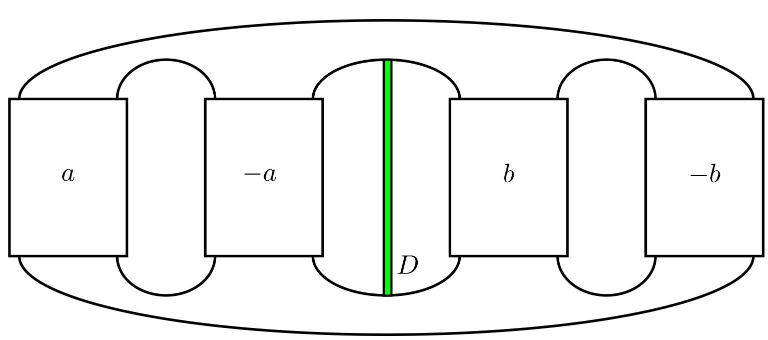



Figure 2.10: **A band move on $P(a, -a, b, -b)$.**

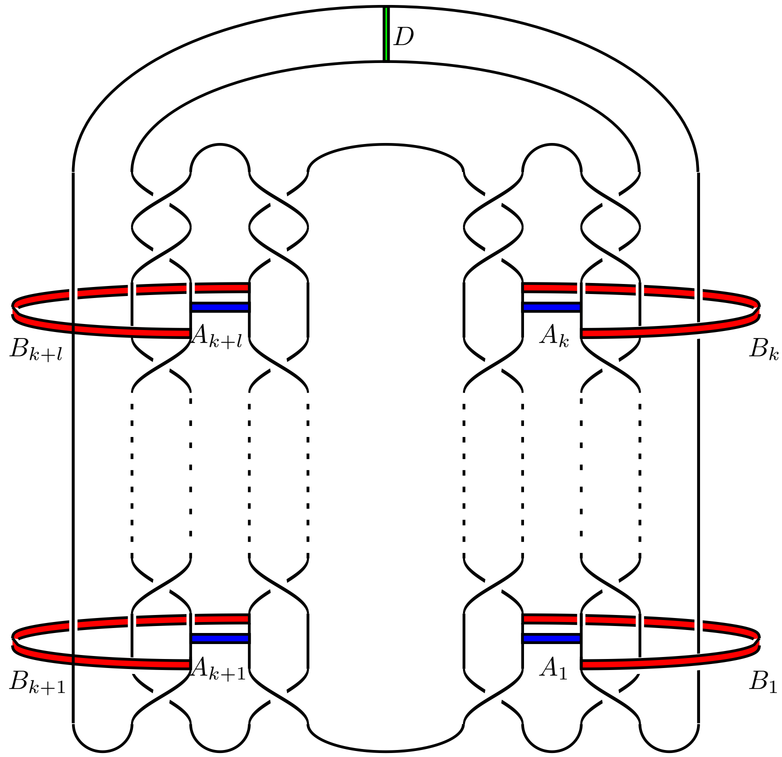



Figure 2.11: **Band moves on** $P(a, -a, b, -b)$**.**

Setting $D = B_0$ gives the result, arguing as in Proposition 2.13 above. Figure 2.12 just shows the bands $A_i$ and $D$. Note that after the band moves given by $A_1$ and $A_{k+1}$ all of the crossings can be removed and it is easy to see that band $D$ simply connects two components together.

□

The proofs of these results can be generalised in two ways. For instance, we may consider pretzel links with more strands and draw similar pictures with more bands. Figure 2.2 can be extended to show that $P(a, -a, a, \dots, \pm a)$ is doubly slice whilst Proposition 2.14 generalises to show that $P(a, -a, b, -b, c, -c)$ is doubly slice when $a, b$ and $c$ are all odd. Alternatively, we can consider Montesinos links by replacing the pairs of twisted strands by rational tangles.

To do these, we use the following generalisation of Lemma 2.12, which also gives a (larger) special case of Zeeman's theorem.

**Lemma 2.15.** *Let $K$ be a 2-bridge knot. Then we can find band moves describing an unknotted sphere with a cross-section given by $K \# -K$.*

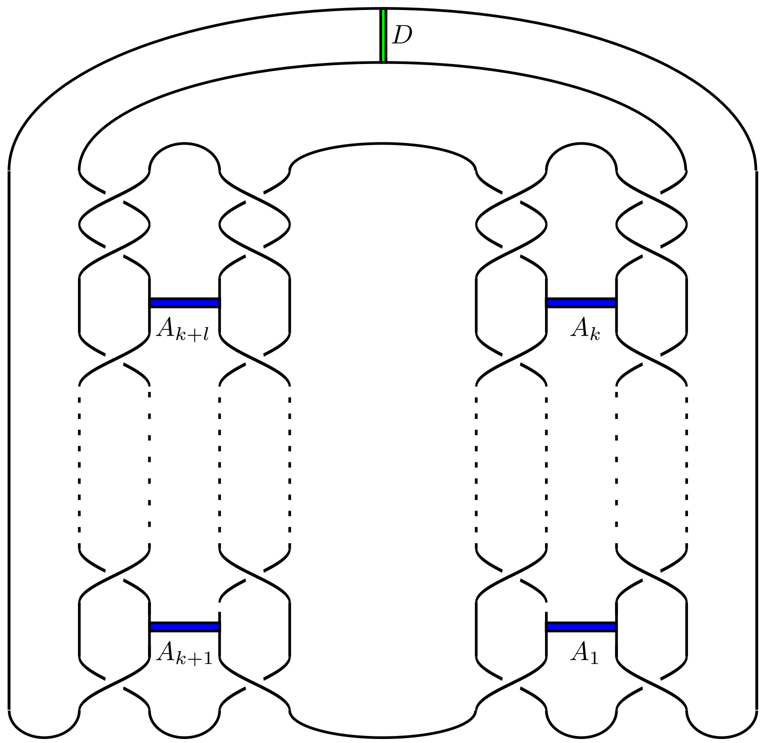



Figure 2.12: **Band moves $D$ and $A_i$.**

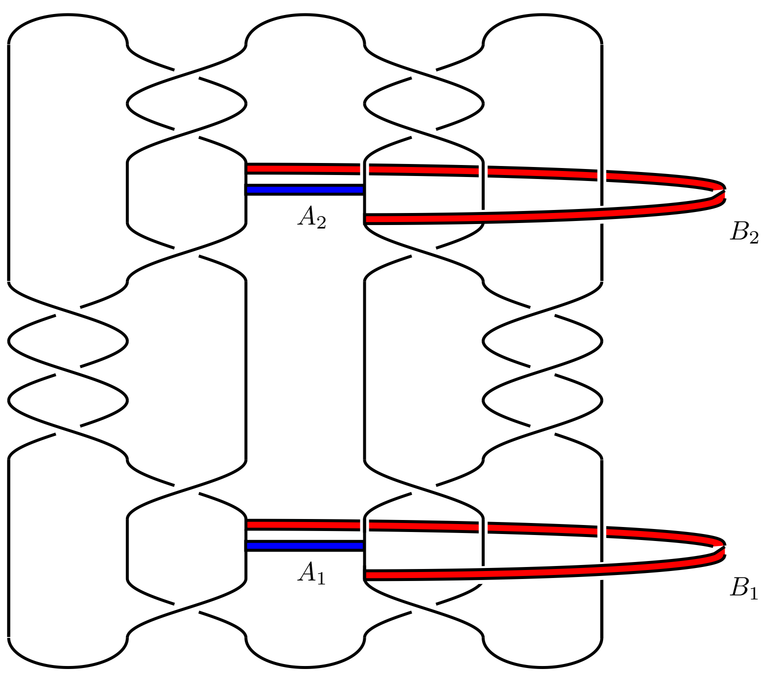



Figure 2.13: **Band moves showing that $S(23,7)\# -S(23,7)$ is doubly slice.**

A concrete example is given in Figure 2.13 for $K = S(-23, 7)$.

*Proof.* Suppose $K = S(p, q)$ with $p > q > 0$. We can find a continued fraction expansion

$$\frac{p}{q} = [a_1, \ldots, a_n]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

where $n$ is odd and $a_{2i-1}$ is even and positive for $1 \leq 2i - 1 < n$. This may be obtained algorithmically: take $a_1$ to be the smallest even integer larger than $\frac{p}{q}$ and choose $a_2 \geq 0$ so that

$$\frac{p}{q} = a_1 - \frac{1}{a_2 - \frac{s}{r}} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\frac{r}{s}}}$$

with $r \geq s > 0$. Repeating this process gives a suitable expansion.

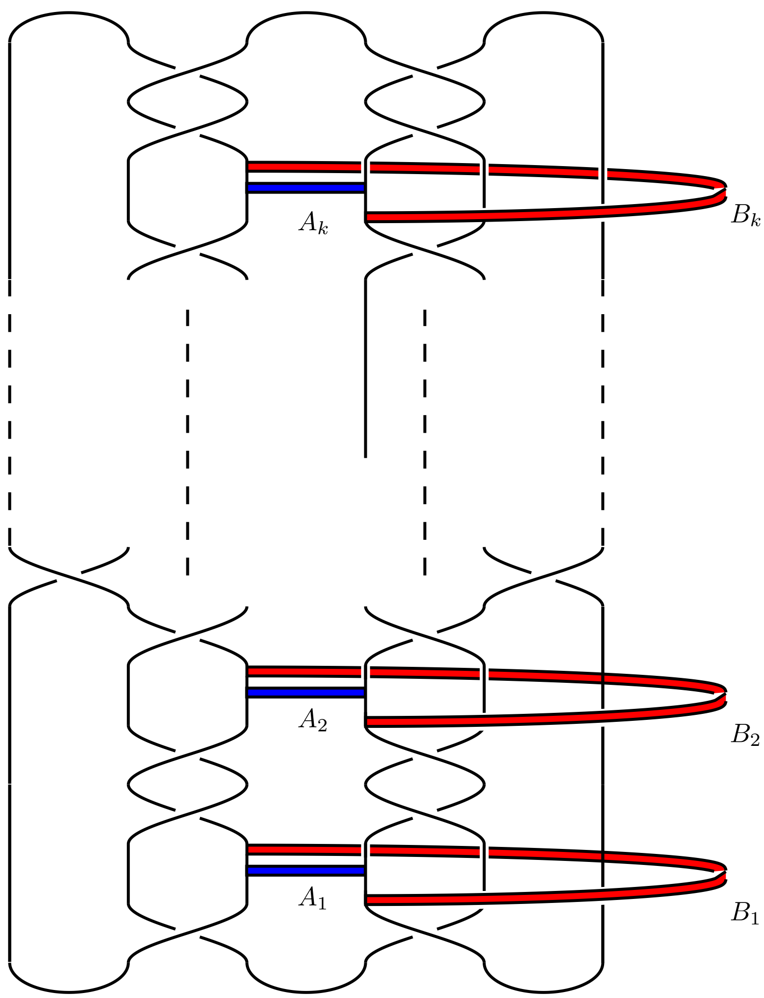



Figure 2.14: **Band moves on a connected sum of 2-bridge knots.**

We obtain a diagram for $K\#-K$ as in Figure 2.14. In this diagram, we have assumed that $a_1 = 4$ but it should be clear how to modify it in the general case. Since $a_{2i-1}$ is even for $2i-1 < n$, the twists in the centre of the diagram appear in pairs, apart from those at the top. We now check that the bands satisfy the conditions of Corollary 2.8. As in the proof of Lemma 2.12, the effect of the band moves $A_1$ and $B_1$ is to change the crossings immediately above the pair of bands. If $a_1 = 2l$, then after $l$ pairs of band moves, there will be an isotopy removing all of these $2l$ crossings in each summand. After band moves have removed all of these crossings, the $a_2$ and $-a_2$ crossings can be untwisted immediately. Continuing, we see that the effect of all of the band moves together is an unknot.

The remaining conditions in Corollary 2.8 can be verified in a similar way, where we use a symmetry between the $A_i$ and $B_i$ bands as in the proof of Lemma 2.12.

□

We can use this to obtain a generalisation of Proposition 2.14.

**Proposition 2.16.** *Let $\{(p_i, q_i)\}_{i=1}^n$ be a finite collection of pairs of coprime integers with $p_i > q_i > 0$ and $p_i$ odd for each $i$. The Montesinos link*

$$M(0; (p_1, q_1), (-p_1, q_1), \ldots, (p_n, q_n), (-p_n, q_n))$$

*is doubly slice.*

*Proof.* This link is obtained from a band move $E$ on $\#_{i=1}^n(S(p_i, q_i)\# - S(p_i, q_i))$ shown in Figure 2.15. Contrary to previous figures, this diagram should be interpreted as the top part of a diagram for a connected sum of 2-bridge knots drawn so that $a_1^i$ is at the top of the picture.

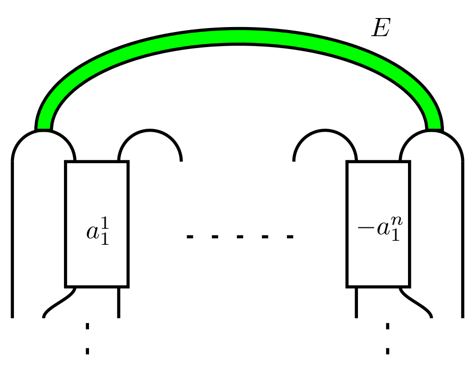



Figure 2.15: **Band $E$ on a sum of 2-bridge knots.**

By taking a connected sum of diagrams and associated band moves of the type described in Lemma 2.15 above we get a similar diagram for this connected sum. To prove this result, we just need to see that band $E$ fits into this picture. As in Proposition 2.14, we need to verify that after we do all the $A_i$ band moves, we can add in band $E$ and get another unlink with an extra component. Between the attaching regions of $E$, the component to which it is attached runs along $A_i$ bands at the top of the diagram. This strand can be extracted easily, giving the diagram in Figure 2.16, from which the result follows.

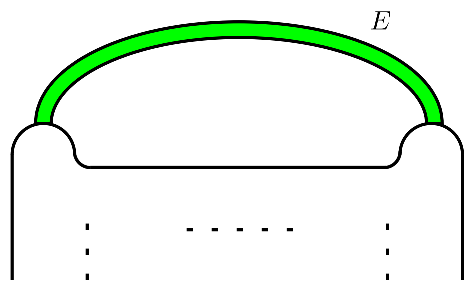



Figure 2.16: **Band $E$ increase the number of components in an unlink.**

□

Taking the double branched cover, Lemma 2.4 shows that when each $p_i$ is odd, we get a smooth embedding of $Y(0; (p_1, q_1), (-p_1, q_1), \ldots, (p_n, q_n), (-p_n, q_n))$ in $S^4$, recovering a result of [CH98]. In fact, the proof of Proposition 2.16 establishes the following.

**Corollary 2.17.** *Let $S = \{(p_i, q_i), (-p_i, q_i)\}_{i=1}^n$ be a finite collection of pairs of coprime integers with $p_i > q_i > 0$ and $p_i$ odd for each $i$. The 2-handle cobordism*

$$W_{S,0} : \#_{i=1}^n (L(p_i, q_i)\# - L(p_i, q_i)) \to Y(S^2; 0; (p_1, q_1), (-p_1, q_1), \ldots, (p_n, q_n), (-p_n, q_n))$$

*embeds smoothly in $S^4$.*

*Proof.* The band move $E$ in the above proof gives a link cobordism $W_L$ between a Montesinos link and a connected sum of 2-bridge links and it is embedded in a trivial sphere. The double branched cover of a slice $S^3 \times I$ intersecting the trivial sphere in $W_L$, with branch set $W_L$, is $W_{S,0}$ and it embeds in $S^4$ as the preimage of this slice. □

### 2.1.1 Aside about doubly slice pretzel links

The focus here is on embedding 3-manifolds in $S^4$ but we could consider the closely related question of which (pretzel) links are doubly slice. This involves a couple of additional complications, namely mutation and the orientation of the links. When we look at branched covers both of these become irrelevant but we can illustrate how they affect the question on the link level with a few examples.

A link $L$ in $S^3$ is said to be (smoothly) *sphere-slice* if it a cross-section of a (smoothly) embedded sphere in $S^4$. Every doubly slice link is sphere-slice.

**Example 2.18.** The pretzel link $P(2, 3, -2, -3)$ is not doubly slice but, since it is a mutant of the doubly slice link $P(2, -2, 3, -3)$, the double branched cover embeds in $S^4$.

These are two-component links, so if they are sphere-slice they must bound a pair of disks (and an annulus). In particular, both components must be slice. The components of $P(2, 3, -2, -3)$ are both trefoil knots and so this link is not sphere-slice.

However, in some cases mutation does not have an effect.

**Example 2.19.** The pretzel links $P(3, -3, 3, -3)$ and $P(3, 3, -3, -3)$ are both doubly slice.

The first of these was seen already in Corollary 2.11, while the latter is shown by Figure 2.17. This figure generalises in the obvious way to $P(a, a, -a, -a)$ for $a \in \mathbb{Z}$. Similar pictures also exists for many mutants of $P(a, -a, a, \ldots, \pm a)$.

The linking numbers between the components give an obstruction to a link being sphere-slice (see [Sat98] for example) which is sensitive to the orientation of the link. The link orientation is not crucial to the double branched cover but it does affect higher branched covers. Every doubly slice link has a quasi-orientation (an orientation defined up to an overall reversal) induced by an orientation of the unknotted sphere. For the diagrams above, this quasi-orientation can be determined by choosing an orientation on the central unknot and requiring that every band respects it.

**Example 2.20.** A link is unlikely to be doubly slice for every orientation. Orienting $P(2, -2, 2)$ so that each of the components is oriented in a clockwise direction is not orientedly doubly slice. The linking number between any pair of components is 2. If the link was sphere-slice with this orientation, it would bound an annulus and a disk and the boundary of the disk component would have total linking number zero with the rest of the link.

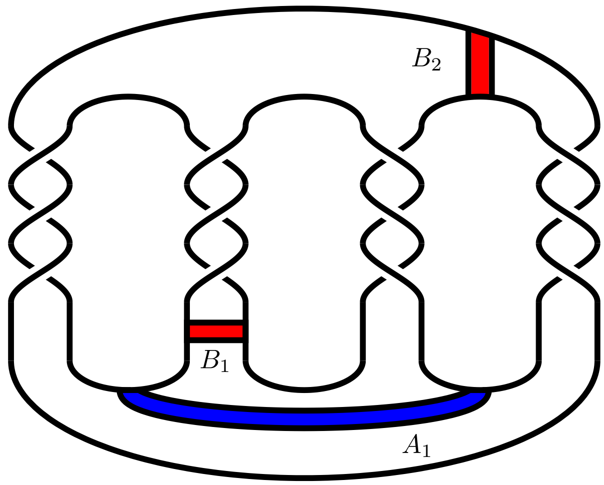



Figure 2.17: **Band moves on** $P(3, 3, -3, -3)$**.**

## 2.2 Constructing embeddings using Kirby diagrams

A Kirby diagram for $S^4$ gives a handle decomposition. Taking only some of these handles gives a 4-dimensional submanifold of $S^4$. We can find an embedding of a 3-manifold in $S^4$ from a sufficiently complicated Kirby diagram by taking the boundary of such a submanifold. Indeed, in principle, every 3-manifold which can be smoothly embedded in $S^4$ can be found in this way.

Figure 2.18 shows a Kirby diagram for $S^4$ with a cancelling pair. The boundary of the 1-handle drawn in black – $S^1 \times S^2$ – embeds smoothly. Figure 2.19 shows an embedding of the 3-torus $T^3$. The boundary of the 1-handles and the 2-handle drawn in black is $T^3$. The blue 2-handles cancel the 1-handles – any other curve linking the 1-handles can be unlinked from it by sliding over the blue meridian – and leave a cancelling 2 and 3-handle.

**Remark 2.21.** Fox [Fox72] showed that $T^3$ does not arise as the double branched cover of a link. The method of producing embeddings from doubly slice links does not find an embedding of $T^3$.

The embeddings given by doubly slice links can be reproduced. The band moves in a diagram for a doubly slice link describe the handle decomposition of an unknotted sphere and we may draw the double branched cover of $S^4$ with this sphere as the branch set. An algorithm for drawing (double) branched covers over surfaces is described in [AK80].

**Example 2.22.** Figure 2.20 gives a picture of $S^4$ with the double branched cover of $L_{a,n}$

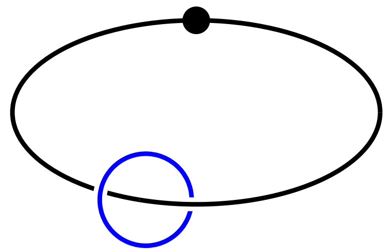



Figure 2.18: **Kirby diagram for $S^4$ with embedded $S^1 \times S^2$.**

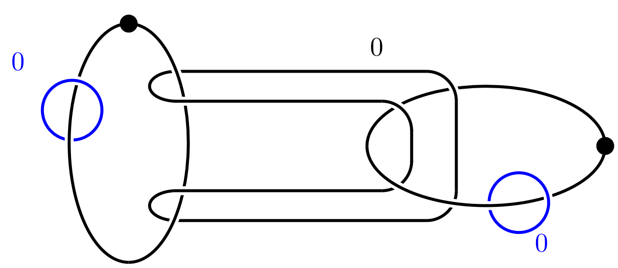



Figure 2.19: **Kirby diagram for $S^4$ with embedded $T^3$.**

as a submanifold. The $\pm a$ twists in this figure are full twists. The double branched cover of $L_{a,n}$ has a surgery diagram as shown in Figure 2.21. Sliding the 2-handles over each other, exchanging some of the curves for 1-handles and cancelling gives the diagram consisting of the 1-handles and black 2-handles in Figure 2.20. The full diagram gives $S^4$ which can be seen by sliding the 1-handle on the right over the 1-handle on the left. The left 1-handle then cancels with the blue 2-handle; the right 1-handle cancels with the $n$-framed 2-handle and the remaining 2-handle with the 3-handle.

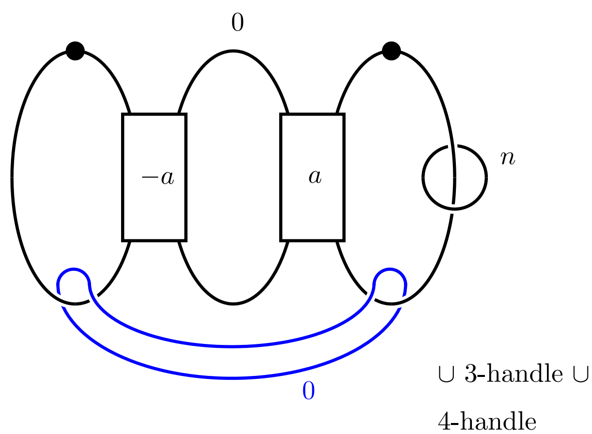



Figure 2.20: **Kirby diagram for $S^4$ with embedded $\Sigma_2(L_{a,n})$.**

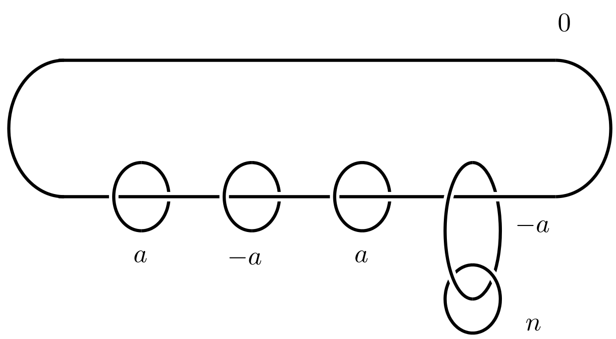



Figure 2.21: **A surgery diagram for $\Sigma_2(L_{a,n})$.**

Next we describe a couple of 'indirect' embeddings. We use Kirby calculus to obtain explicit embeddings in some cobordisms and then argue that these cobordisms embed in $S^4$.

**Lemma 2.23.** *Let $Y = Y(\Sigma_g; r; (a_1, b_1), \ldots, (a_n, b_n))$. There is a smooth embedding of*

*$Y(\Sigma_{g+1}; r; (a_1, b_1), \ldots, (a_n, b_n))$ in $Y \times I$.*

*Proof.* Figure 2.22 shows a modification of a Kirby diagram with boundary $Y$, where all of the handles are attached near to the central curve. This describes the product cobordism since we add a cancelling 1-2-handle pair, then cancelling 2 and 3-handles. Attaching the 2-handles in the other order, we see the specified 3-manifold – compare to Figure 1.8, which illustrates how to increase the base genus.

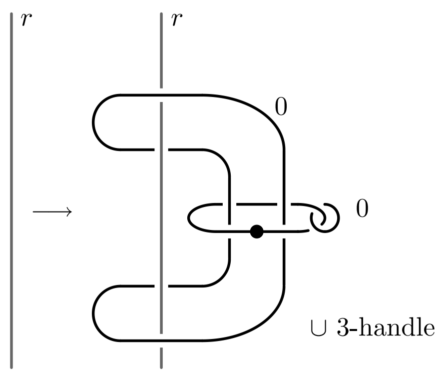



Figure 2.22: **Adding to the base genus.**

□

**Corollary 2.24.** *If $Y(S^2; r; (a_1, b_1), \ldots, (a_n, b_n))$ embeds smoothly in $S^4$ then so does $Y(\Sigma_{g+1}; r; (a_1, b_1), \ldots, (a_n, b_n))$ for any $g > 0$.*

**Remark 2.25.** This is a result of Crisp and Hillman [CH98, Lemma 3.2].

**Lemma 2.26.** *Let $S = \{(a_i, b_i)\}$ be a set of Seifert invariants. We get a sum of lens spaces and a 2-handle cobordism $W_{S,r}$ to the Seifert manifold $Y(S^2; r; S)$. There is a smooth embedding of $Y(N_g; t; S)$ in $W_{S,r}$ for $t = r + 2g - 4n$ when $0 \leq n \leq g$.*

*Proof.* Figure 2.23 shows a relative Kirby diagram for $W_{S,r}$. Starting with the handles drawn in red, we can add the black 1-handles and the green and blue 2-handles. The blue and black handles cancel so this is the same cobordism as the one obtained by just adding the green handle.

The boundary of the manifold given by the red, black and green handles is $Y(N_g; t; S)$ for some $t$. There are $g$ 1-handles which can have either positive or negative linking with the green 2-handle – the two drawn in Figure 2.23 have opposite linking. Thus $t$ lies

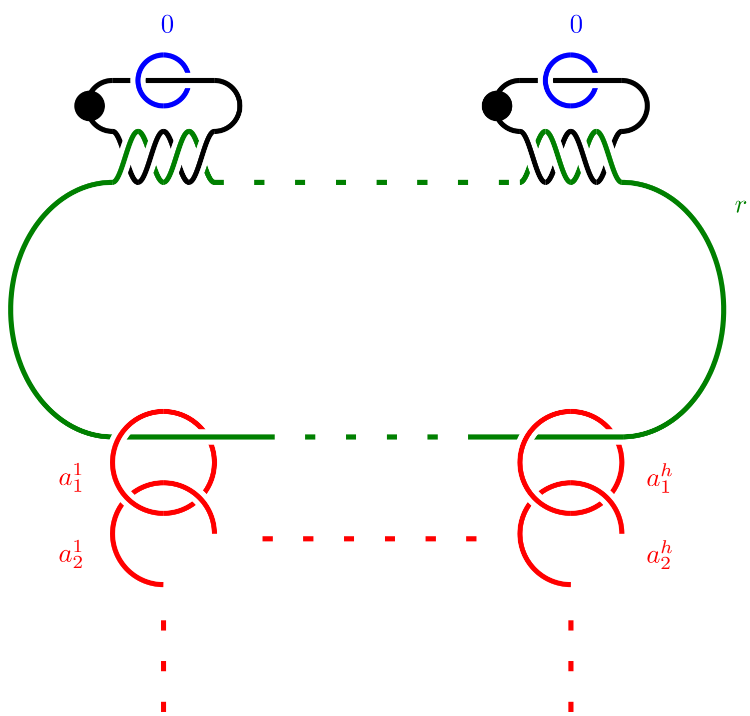



Figure 2.23: **Seifert manifolds with non-orientable base embed in a cobordism between a connected sum of lens spaces and a Seifert manifold with base $S^2$.**

between $r-2g$ and $r+2g$ and takes values $r+2g-4n$ where $n$ is the number of 1-handles with negative linking. □

We can use this to reprove part of [CH98, Proposition 1.2].

**Corollary 2.27.** *If $S = \{(a_i, b_i), (a_i, -b_i)\}$ is a set of complementary pairs of odd Seifert invariants then $Y = Y(N_g; t; S)$ embeds smoothly in $S^4$ for $t = 2g - 4n$ for $0 \leq n \leq g$.*

*Proof.* In this case the 2-handle cobordism $W_{S,0}$ embeds smoothly in $S^4$ by Corollary 2.17. By the previous lemma, $Y$ embeds in this cobordism. □

Crisp-Hillman prove the stronger statement that, for sets of Seifert invariants of this form, these are the only values of $t$ for which $Y(N_g; t; S)$ embeds smoothly.

We will now use Kirby calculus to construct an embedding for another Seifert manifold.

**Lemma 2.28.** *Suppose $Y$ is the boundary of a Kirby diagram consisting of 4 2-handles. Suppose these are attached along framed knots $\gamma_i$ $(1 \leq i \leq 4)$ with the following properties:*

- *The sublink given by $\gamma_1$ and $\gamma_2$ is a 0-framed unlink;*
- *the sublink given by $\gamma_1$ and $\gamma_3$ is a 0-framed unlink;*
- *the linking number of $\gamma_1$ and $\gamma_4$ is $\pm 1$.*

*Then $Y$ embeds smoothly in $S^4$.*

*Proof.* We can draw a Kirby diagram as follows. Exchange $\gamma_1$ and $\gamma_2$ for 1-handles and add 0-framed meridians to $\gamma_2$ and $\gamma_4$.

Then $\gamma_2$ and its 0-framed meridian give a cancelling pair – whenever a 2-handle crosses over $\gamma_2$ in the diagram we may change this to an undercrossing by sliding the other component over the meridian. This pair can therefore be removed.

Similarly, we can remove every crossing of $\gamma_3$ over $\gamma_4$. Since it is 0-framed and can be drawn such that it has no crossings with $\gamma_1$, we may add a cancelling 3-handle.

Our diagram now consists of a 1-handle attached along $\gamma_1$, $\gamma_4$ and a 0-framed meridian of $\gamma_4$. By sliding $\gamma_4$ over this meridian, we may change any crossing of $\gamma_4$ with itself. Since the linking number of $\gamma_1$ with $\gamma_4$ is $\pm 1$ we see that they give a canceling pair. After removing them, we may add a 3-handle and a 4-handle to get the standard Kirby diagram of $S^4$.

It then follows that $Y$ is the boundary of a smooth submanifold of $S^4$. □

**Remark 2.29.** Statements similar to Lemma 2.28, for example about diagrams with more 2-handles, can be established by much the same argument.

We use Lemma 2.28 to describe another embedding.

**Example 2.30.** The Seifert manifold $Y(S^2; 0; (4,1), (4,1), (12,-7))$ embeds smoothly in $S^4$.

We rewrite this Seifert manifold as $Y(S^2; 1; (4,1), (4,1), (12,5))$. Using the continued fraction $12/5 = [2,-3,-2]^-$ this is the boundary of the plumbing shown in Figure 2.24. We blow down the +1-framed curve to get the first picture in Figure 2.25 and then perform the indicated Kirby moves.

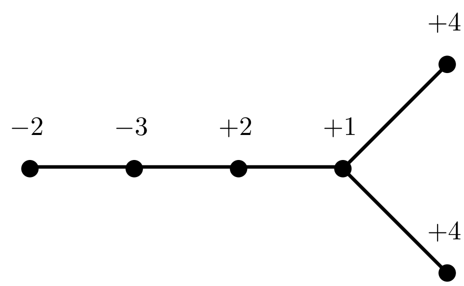
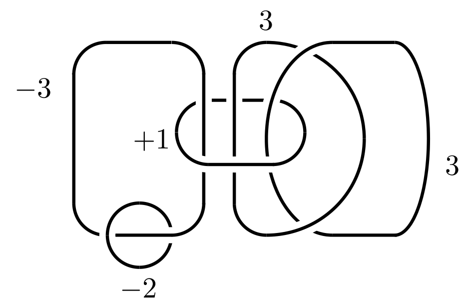



Figure 2.24: **Plumbing graph for** $Y(S^2; 1; (4,1), (4,1), (12,5))$**.**

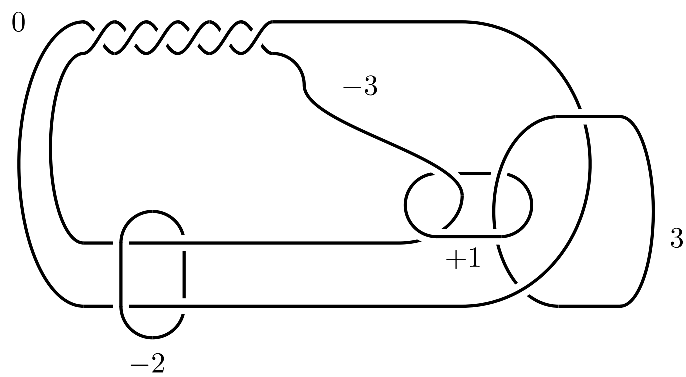
The second diagram has a 0-framed unknot which we think of as $\gamma_2$ to fit in with the notation of Lemma 2.28. The final picture is a Kirby diagram for a 4-manifold $X$ to which the lemma applies but another three handle slides are needed to draw it in the required form. The bands determining these slides are drawn. Note that there is another 0-framed unknot which we call $\beta$ and should think of as $\gamma_1 + \gamma_2$. It forms a 0-framed unlink with $\gamma_2$.

The next handle slide uses band $A$ to slide the curve with framing 2 over the one with framing $-4$, to get a 0-framed curve $\gamma_3$. We then slide the $-4$ framed curve over $\beta$ using band $B$ to get $\gamma_4$ and finally use band $C$ to slide $\beta$ over $\gamma_2$. This gives a 0-framed curve $\gamma_1$.

It is easy to check that the sublink given by $\gamma_1$ and $\gamma_2$ is a 0-framed unlink. The linking number of $\gamma_1$ and $\gamma_4$ is a homological property of $X$ and can be computed using the intersection form of $X$. A matrix for the form can be found using the linking numbers in the final diagram in Figure 2.25 and a simple calculation verifies that $\gamma_1$ and $\gamma_4$ have linking number $\pm 1$.




↓ Slide a handle with framing +3 over the handle with framing −3.




↓ Blow down, handle slide.

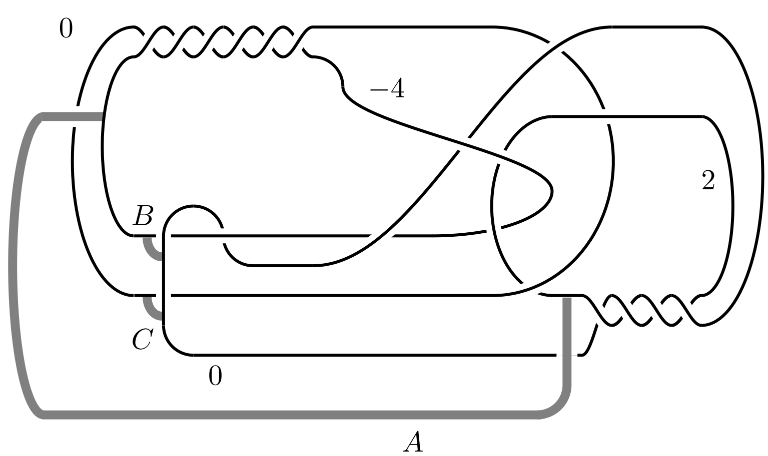



Figure 2.25: **Kirby moves.**

Both $\gamma_1$ and $\gamma_3$ are 0-framed and we can see the sublink consisting of these two curves by band summing the components in the last picture of Figure 2.25 along bands $A$ and $C$. This gives an unlink, shown in Figure 2.26, and so Lemma 2.28 gives an embedding.

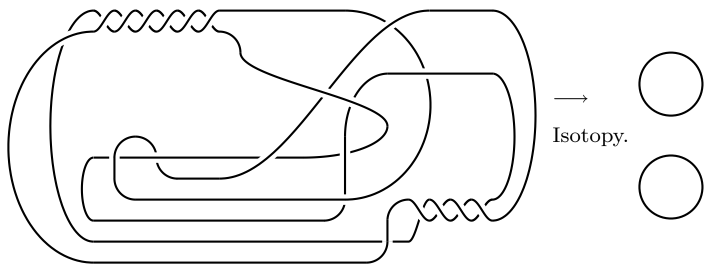



Figure 2.26: $\gamma_1$ **and** $\gamma_3$ **give a 2-component unlink.**

□

### 2.2.1 Fundamental groups of submanifolds of $S^4$

We can also use Kirby calculus to give a quick proof that every finitely presented group is the fundamental group of a smooth submanifold of $S^4$. This was shown by Dranĩsnikov-Repovŝ [DsRs93] as a corollary to a result of Stallings [Sta65].

**Theorem 2.31.** *Let $G$ be a finitely presented group. Then $G$ is the fundamental group of a 4-manifold with boundary which is a smooth submanifold of $S^4$.*

*Proof.* We may draw a Kirby diagram as follows. Take a finite presentation of $G$ and a 1-handle for each generator. Then add a 0-framed 2-handle tracing out each relation to get a manifold whose fundamental group is $G$. We may arrange that the 2-handles alone form a 0-framed unlink by changing crossings in their attaching link if necessary. This does not affect the fundamental group but it gives a manifold which embeds in $S^4$ – adding a 0-framed meridian to each 1-handle, a 3-handle for each relation and a 4-handle gives a diagram for $S^4$. □

**Remark 2.32.** This is a variant of a standard argument that $G$ is the fundamental group of a smooth 4-manifold (see [GS99, Exercise 4.6.4(b)] for example).

**Remark 2.33.** From a Kirby diagram of this form for a manifold $U$ – with the 2-handles forming a 0-framed unlink and no 3 or 4-handles – we can easily find a diagram for $S^4 \backslash U$

by exchanging all of the 1-handles for 0-framed 2-handles and the 2-handles for 1-handles. Consequently, we can easily find a presentation for $\pi_1(S^4 \backslash U)$.

# Chapter 3

# Obstructions to a 3-manifold embedding in $S^4$

Our approach can be summarised as follows. A 3-manifold $Y$ which embeds smoothly in $S^4$ results in a pair of submanifolds of $S^4$ with boundary $Y$. These necessarily have 'simple' rational homology primarily determined by that of $Y$. By gluing these pieces to a known 4-manifold $X$ with boundary $Y$ we obtain smooth, closed 4-manifolds. Many of the properties of this 4-manifold are largely determined by those of $X$. Obstructions are obtained by arranging $X$ so that the hypothesised 4-manifold has impermissible properties. The obstruction techniques we use are largely based on this idea.

In Section 3.2 we use Donaldson's Theorem A on the intersection forms of smooth 4-manifolds while in Section 3.4 we describe obstructions based on Furuta's 10/8 theorem and on Heegaard-Floer homology.

## 3.1 Generalities

First we collect some useful generalities which are needed in both cases.

**Lemma 3.1.** *Suppose a 3-manifold $Y$ embeds smoothly in $S^4$. Then there is a splitting $S^4 = U_1 \cup_Y -U_2$ for smooth 4-manifolds $U_i$ with boundary $Y$ such that*

1. $H^2(Y;\mathbb{Z}) \cong H^2(U_1;\mathbb{Z}) \oplus H^2(U_2;\mathbb{Z})$;

2. $H^2(U_i;\mathbb{Z}) \cong H_1(U_j;\mathbb{Z})$ *for* $i \neq j$;

3. $b_3(U_i) = 0$;

*4.* $\sigma(U_i) = 0$.

Note in particular that the torsion subgroup $\tau H^2(Y) \cong G \oplus G$ where $G$ is the (common) torsion subgroup of $H^2(U_1)$ and $H^2(U_2)$.

*Proof.* The first three statements follow by applying the Mayer-Vietoris sequence to this decomposition of $S^4$ and Alexander duality. Since $b_3(U_i) = 0$, it follows from the exact cohomology sequence of the pair $(U_i, Y)$ that $b_1(U_i) + b_2^0(U_i) = b_1(Y)$, where $b_2^0(U_i)$ denotes the rank of the kernel of the restriction map $H^2(U_i, Y) \to H^2(U_i)$. This implies that $b_2(U_i) = b_2^0(U_i)$ and, in particular, that the signature is zero. □

In particular, when $Y$ is a rational homology sphere, $U_1$ and $U_2$ are both rational balls and when $b_1(Y) = 1$, $U_1$ and $U_2$ have the rational homology of $S^1 \times D^3$ and $S^2 \times D^2$.

We briefly mention that $H^2(Y;\mathbb{Z})$ classifies the spin$^c$ structures on $Y$. We will return to this in Section 3.4. In a similar way, the spin structures on $Y$ are a torsor for $H^1(Y;\mathbb{Z}/2)$ so it is useful to record the following consequences of the Mayer-Vietoris sequence and Alexander duality.

**Corollary 3.2.** *If $Y$ embeds in $S^4$ giving a splitting as $S^4 = U_1 \cup_Y -U_2$, there are isomorphisms*

$$H^1(Y;\mathbb{Z}/2) \cong H^1(U_1;\mathbb{Z}/2) \oplus H^1(U_2;\mathbb{Z}/2)$$

*and*

$$H^1(U_i;\mathbb{Z}/2) \cong H_2(U_j;\mathbb{Z}/2) \text{ for } i \neq j.$$

□

We now describe the homological properties of $X \cup_Y U_i$ for 'suitable' 4-manifolds $X$.

**Proposition 3.3.** *Let $Y$ be a 3-manifold which bounds 4-manifolds $U, X$ where $U$ is a submanifold of $S^4$ and $H^3(X) = 0$. Let $W = X \cup_Y -U$ and let $K$ be the kernel of the inclusion map*

$$H_1(Y;\mathbb{Z}) \to H_1(X;\mathbb{Z})$$

*and denote its rank by $k$.*

*If the image of $K$ in $H_1(U;\mathbb{Z})$ also has rank $k$ then $b_2(W) = b_2(X) - k$ and $\sigma(W) = \sigma(X)$.*

*Proof.* We may calculate $b_1(W)$ using the Mayer-Vietoris sequence. The condition on $K$ implies that the three first homology terms give a short exact sequence and so $b_1(W) = b_1(X) + b_1(U) - b_1(Y)$.

Computing the Euler characteristic of $W$ gives an expression for $b_2(W)$ which may be reduced to the claimed form using the equations $b_1(U)+b_2(U) = b_1(Y)$ and $b_1(X)-b_1(Y) = -k$. These follow from Lemma 3.1 and the condition on $X$.

The signatures of $W$ and $X$ are equal as $\sigma(U) = 0$ by Lemma 3.1. □

**Remark 3.4.** When $k = 0$, the condition on the rank of $K$ in $H_1(U;\mathbb{Z})$ holds trivially, while for $k = 1$, it will hold for at least one of $U$ and $S^4\backslash U$.

## 3.2 Diagonalisation

We will use Donaldson's theorem about 4-manifolds with definite intersection forms to obtain an obstruction.

**Theorem 3.5** (Donaldson [Don87])**.** *If $W$ is a closed, oriented, smooth 4-manifold and the intersection form $Q_W : H_2(W;\mathbb{Z}) \otimes H_2(W;\mathbb{Z}) \to \mathbb{Z}$ is negative definite then $Q_W$ is diagonalisable.*

The first objective is to develop a general obstruction in the case where $X$ is such that Proposition 3.3 gives a closed definite 4-manifold.

The obstruction will be used to prove Theorems 3.16, 3.17 and 3.32. To that end, we will find appropriate 4-manifolds with the boundaries required. This produces a combinatorial condition as an embedding obstruction. In Section 3.3 this is analysed to obtain the results.

Recall that $W = X \cup_Y -U$ where $U$ is a submanifold of $S^4$. If $X$ is chosen so that $b_2(X) - k = -\sigma(X)$ then $W$ is negative definite and Donaldson's theorem applies to show that the intersection form of $W$ is diagonal. Let $\{e_i\}$ be a basis for $H_2(W)/\text{Torsion}$ such that $e_i \cdot e_j = -\delta_{ij}$. Next we consider the induced map $\iota_* : H_2(X) \to H_2(W)$.

We may choose a basis $\{h_1, \dots h_n\}$ of $H_2(X) \cong \mathbb{Z}^n$. Let $Q_X$ denote the matrix of the intersection pairing with respect to this basis.

Following [Lis07a] we can use these to define a 'subset'.

**Definition 3.6.** *Let $v_i = \iota_*(h_i) \in H_2(W)/\text{Torsion}$ for each $1 \leq i \leq n$. We call $S = \{v_1, \dots v_n\}$ the subset associated to the pair $(X, U)$.*

*The matrix $A(S) = [e_i \cdot v_j]$ is called the matrix of $S$.*

Clearly, $S$ and $A(S)$ just give different ways of recording the same information. We will switch between the two freely whenever it is convenient. Note that for the bases $\{h_i\}$ and $\{e_j\}$ the map $\iota_*$ is represented by the matrix $A(S)^t$.

An important feature of the subset $S$ is that it encodes information about the manifold $X$ and the image of the torsion subgroup $\tau H^2(U)$ of $H^2(U)$ in $H^2(Y)$.

**Lemma 3.7.** *Let $W = X \cup_Y -U$ where $U$ is a smooth 4-dimensional submanifold of $S^4$ and suppose $W$ is negative definite. Choose a basis for $H_2(X)$ and let $S$ be the associated subset.*

*The matrix $A(S)$ is such that $Q_X = -A(S)A(S)^t$.*

*Proof.* The homology classes in $H_2(X)$ are represented by embedded surfaces and the intersection form counts the signed intersection points of these surfaces. If surfaces $\{\alpha_i\}$ represent classes in $H_2(X)$ then the same surfaces sit inside $W$ to represent the images of these homology classes under the inclusion induced map. Since the intersection points are the same, $Q_X(h_i, h_j) = -v_i \cdot v_j$. The matrix $A(S)^t$ represents the inclusion map so $v_i = A(S)^t h_i$ and so for every pair $h_i, h_j$, $Q_X(h_i, h_j) = -A(S)A(S)^t(h_i, h_j)$. □

**Theorem 3.8.** *Let $U$ be a submanifold of $S^4$ and $X$ be such that $H^3(X;\mathbb{Z}) = 0$, $H_1(X;\mathbb{Z})$ is torsion-free and the matrix $Q_X$ is non-singular. Suppose $\partial X = -\partial U = Y$ and that $W = X \cup_Y U$ is negative definite. Let $S$ be the associated subset.*

*There is an isomorphism between the torsion subgroup of the image of the restriction map $H^2(U) \to H^2(Y)$ and $\left(\frac{\operatorname{im} A(S)}{\operatorname{im} Q_X}\right)$ and this is facilitated by the inclusion induced map $\delta : H^2(X) \to H^2(Y)$.*

*Proof.* We follow the approach of [GJ11, Proposition 2.5]. Consider the following diagram with the maps of cohomology induced by the inclusion $(X, Y) \hookrightarrow (W, U)$:

$$\begin{array}{ccccccccc}
\longrightarrow & H^2(W,U) & \xrightarrow{\alpha} & H^2(W) & \xrightarrow{\beta} & H^2(U) & \longrightarrow & H^3(W,U) \\
& \iota_1 \cong \downarrow & & \iota_2 \downarrow & & \iota_3 \downarrow & & \iota_4 \cong \downarrow \\
\longrightarrow & H^2(X,Y) & \xrightarrow{\gamma} & H^2(X) & \xrightarrow{\delta} & H^2(Y) & \longrightarrow & H^3(X,Y).
\end{array}$$

The rows of this diagram are exact and it is commutative since all of the maps are given by restriction.

Given the basis $\{h_i\}$ for $H_2(X)$ we may choose the dual and Poincaré dual bases for $H^2(X)$ and $H^2(X, Y)$. With these choices the map $\gamma$ is represented by $Q_X$. This sets up an identification of a subgroup of $H^2(Y)$ with $\operatorname{coker} Q_X$ via $\delta$. Since $\det Q_X \neq 0$, this lies in the torsion subgroup of $H^2(Y)$ and the fact that $H_1(X)$ is torsion-free shows that this gives the whole torsion subgroup.

We are interested in the image of $\iota_3$. This has a subgroup given by the image of $\iota_3 \circ \beta$. Since this is the same as the image of $\delta \circ \iota_2$ it is a finite group.

We may choose the dual basis to $\{e_i\}$ for $H^2(W)/\text{Torsion}$ so that the map $\iota_2$ is represented by the matrix $A(S)$. Note that since $H^2(X)$ is free abelian any torsion in $H^2(W)$ must map trivially.

The image of $\delta \circ \iota_2$ is therefore isomorphic to $\left(\frac{\operatorname{im} A(S)}{\operatorname{im} Q_X}\right)$.

To see that this gives the entire torsion subgroup of the image of $\iota_3$, we compare the orders. By Lemma 3.7, $Q_X = -A(S)A(S)^t$ and so the order of this subgroup is $|\det A(S)|$. By Lemma 3.1 the torsion of the image of $\iota_3$ also has order given by the square root of $|\operatorname{coker} Q_X|$. □

**Remark 3.9.** The assumption in Theorem 3.8 that $U$ is a submanifold of $S^4$ can sometimes be weakened. When $Y$ is a rational homology sphere and $U$ any rational ball this result is Theorem 3.5 of [GJ11].

### 3.2.1 Definite 4-manifolds bounded by Seifert manifolds

We will apply Theorem 3.8 to obtain obstructions to embedding Seifert manifolds. To do this, we describe the relevant negative definite 4-manifolds.

Recall that a negative definite plumbing bounded by the lens space $L(p, q)$ can be constructed by plumbing on a linear graph with weights given by the negative continued fraction. A similar construction works when $Y$ is a Seifert manifold with base $S^2$ and $e(Y) > 0$. It may be arranged that the Seifert invariants are of the form $(a_i, b_i)$ with $a_i > -b_i > 0$. A weighted graph, which yields a plumbing with boundary $Y$, can be obtained by taking a central vertex weighted by the central framing and attaching legs with weights according to the negative continued fractions of $a_i/b_i$. It is shown in [NR78] that this is negative definite whenever $e(Y) > 0$ and negative semi-definite when $e(Y) = 0$.

Recall that to get a surgery picture for a Seifert manifold with a different base surface, we modify the diagram at the central curve. Figure 1.8 shows how to add an orientable handle and Figure 1.9 how to add an $\mathbb{RP}^2$ summand to the base.

The construction of negative definite manifolds with Seifert boundaries can be extended. The intersection forms depend primarily on the Seifert invariants, not the base surface.

**Proposition 3.10.** *Let $Y_F = Y(F; r; (a_1, b_1), \dots (a_n, b_n))$ where $F$ is a closed surface and $L = -\#_{i=1}^n L(a_i, b_i)$.*

*There are 4-manifolds $X_L$ and $X_F$ with boundaries $L$ and $Y_F$ respectively. The 4-manifolds $X_L$ and $X_{S^2}$ are obtained by plumbing and the intersection form $Q_{X_F}$ is equivalent to $Q_{X_{S^2}}$ if $F$ is orientable and to $Q_{X_L}$ otherwise.*

*In addition, $X_L$ can always be chosen to be negative definite and $X_{S^2}$ can be chosen to be negative definite if $e(Y_{S^2}) > 0$ and semi-definite if $e(Y_{S^2}) = 0$.*

*Proof.* The manifolds $X_L$ and $X_{S^2}$ are described above.

We get a Kirby diagram for $X_F$ by modifying the diagram for $X_{S^2}$. We add 1-handles in place of the new 0-framed 2-handles in Figures 1.8 and 1.9.

The intersection forms of these manifolds are easy to describe. There are two cases, depending on whether the base surface is orientable or not, but the intersection form does not depend on the genus. When the base surface $F$ is orientable, there is a basis for $H_2(X; \mathbb{Z})$ given by the cores of the 2-handles. We may orient these so that the intersection form is given by the incidence matrix of the plumbing graph, obtained by ignoring any 1-handles. This gives a manifold $X_F$ with the same intersection form as $X_{S^2}$.

When the base surface is non-orientable, the central curve does not contribute to the second homology. The intersection form is given by the other 2-handles. This is the same as the intersection form of $X_L$. □

We can now apply Theorem 3.8.

**Corollary 3.11.** *Let $Y$ be a connected sum of lens spaces or a Seifert manifold with orientable base orbifold and $e(Y) > 0$, which embeds smoothly in $S^4$ and let $X$ be the negative definite 4-manifold with boundary $Y$ from Proposition 3.10. Then there are $b_2(X) \times b_2(X)$ integer matrices $A_1, A_2$ such that $A_i A_i^t = -Q_X$ for $i = 1, 2$. Viewing $A_1, A_2$ and $Q_X$ as maps $\mathbb{Z}^{b_2(X)} \to \mathbb{Z}^{b_2(X)}$ let $H_i = \frac{\operatorname{im} A_i}{\operatorname{im} Q_X}$ be subgroups of* $\operatorname{coker} Q_X$.

*Then* $\operatorname{coker} Q_X = H_1 \oplus H_2$ *and* $H_1 \cong H_2$.

*Proof.* The embedding produces a splitting $S^4 = U_1 \cup_Y -U_2$. Applying Theorem 3.8 to $W_i = X \cup_Y -U_i$ gives the matrices $A_i$ and identifies the images of the restriction maps $\tau H^2(U_i) \to H^2(Y)$ with $\frac{\operatorname{im} A_i}{\operatorname{im} Q_X}$ using the map $\delta$.

The result can now easily be deduced from Lemma 3.1 since the isomorphism $H^2(U_1) \oplus H^2(U_2) \to H^2(Y)$ is induced by the inclusions of $Y$ into each $U_i$. □

**Remark 3.12.** When $Y$ is an integral homology sphere then $H_1 = H_2 = \{0\}$ and it is possible to just take $A_1 = A_2$.

Otherwise, the condition that $H_1$ and $H_2$ have trivial intersection implies that $A_1$ and $A_2$ must be different. In particular, since $H_i$ is the subgroup of $\operatorname{coker} Q_X$ generated by the columns of $A_i$, the column spaces of the matrices must be different.

**Remark 3.13.** Corollary 3.11 holds for any negative definite 4-manifold $X'$ provided the inclusion map $H_1(Y;\mathbb{Q}) \to H_1(X',\mathbb{Q})$ is an isomorphism.

**Corollary 3.14.** *Let $Y$ be a Seifert manifold with orientable base orbifold and $e(Y) = 0$. If $X$ is the semi-definite 4-manifold with boundary $Y$ from Proposition 3.10 then there is a $b_2(X) \times b_2(X) - 1$ integer matrix $A$ such that $AA^t = -Q_X$.*

*Proof.* The embedding splits $S^4$ as $U_1 \cup_Y -U_2$. The kernel $K$ of the map $H_1(Y;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ has rank one. By Lemma 3.1 the inclusion maps give an isomorphism $H_1(Y;\mathbb{Z}) \cong H_1(U_1;\mathbb{Z}) \oplus H_1(U_2;\mathbb{Z})$ and hence $K$ must map to a rank one subgroup of $H_1(U_i;\mathbb{Z})$ for some $i = 1, 2$. For this $i$, Proposition 3.3 shows that $X \cup_Y -U_i$ is negative definite so the result then follows by applying Lemma 3.7. □

While a Seifert manifold with a non-orientable base surface is also the boundary of a canonical negative definite manifold $X$, the first homology of $X$ is not torsion-free. However, we may modify the proof of Theorem 3.8 to recover a result slightly weaker than Corollary 3.11.

**Corollary 3.15.** *Let $Y$ be a Seifert manifold with non-orientable base orbifold $P_k$, which embeds smoothly in $S^4$ and let $X$ be the negative definite 4-manifold from Proposition 3.10. Then there are $b_2(X) \times b_2(X)$ integer matrices $A_1, A_2$ such that $A_iA_i^t = -Q_X$ for $i = 1, 2$. Viewing $A_1, A_2$ and $Q_X$ as maps $\mathbb{Z}^{b_2(X)} \to \mathbb{Z}^{b_2(X)}$ let $H_i = \frac{\operatorname{im} A_i}{\operatorname{im} Q_X}$ be subgroups of $\operatorname{coker} Q_X$.*

*Then $\operatorname{coker} Q_X \cong H_i \oplus H_i$ for $i = 1, 2$ and $|H_1 \cap H_2| \leq 2$.*

*Proof.* As before $S^4 = U_1 \cup_Y -U_2$ and this gives subsets with associated matrices $A_1$ and $A_2$. Let $t$ be the unique element of order two in $H^2(X) \cong \mathbb{Z}^{b_2(X)} \oplus \mathbb{Z}/2$. The commutative diagram from the proof of Theorem 3.8 can be extended, with $i = 1, 2$.

$$\begin{array}{ccccccccc}
\longrightarrow & H^2(W_i,U_i) & \xrightarrow{\alpha^i} & H^2(W_i) & \xrightarrow{\beta^i} & H^2(U_i) & \longrightarrow & H^3(W_i,U_i) \\
 & \iota_1^i \cong \downarrow & & \iota_2^i \downarrow & & \iota_3^i \downarrow & & \iota_4^i \cong \downarrow \\
\longrightarrow & H^2(X,Y) & \xrightarrow{\gamma} & H^2(X) & \xrightarrow{\delta} & H^2(Y) & \longrightarrow & H^3(X,Y) \\
 & & & q_1 \downarrow & & q_2 \downarrow & & \\
 & & & \dfrac{H^2(X)}{\text{Torsion}} & \xrightarrow{\delta'} & \dfrac{H^2(Y)}{\langle\delta(t)\rangle} & &
\end{array}$$

With respect to the appropriate bases, $q_1 \circ \gamma$ is represented by $Q_X$. Note that the image of this composition is the kernel of $\delta'$ so there is an isomorphism between $\operatorname{coker} Q_X$ and $\operatorname{im} \delta'$. The torsion of $H^2(W_i)$ maps trivially under $q_1 \circ \iota_2^i$ so we can identify this map with the matrix $A_i$. The group $H_i$ can now be seen as the image of $\delta' \circ q_1 \circ \iota_2^i$.

By Proposition 3.10,

$$\operatorname{coker} Q_X = \bigoplus_{i=1}^{n} \mathbb{Z}/a_n,$$

where the $a_i$ come from the Seifert invariants of $Y$. We may order the $a_i$ by writing each as $a_i = 2^{t_i} s_i$ with $s_i$ odd and arranging that $t_1 \geq t_2 \geq \ldots \geq t_n$. With this ordering, [CH98, Lemma 3.4] tells us that

$$\tau H^2(Y) = \left(\bigoplus_{i=3}^{n} \mathbb{Z}/a_i\right) \oplus \mathbb{Z}/2a_1 \oplus \mathbb{Z}/2a_2 \text{ or } \left(\bigoplus_{i=2}^{n} \mathbb{Z}/a_i\right) \oplus \mathbb{Z}/4a_1.$$

By Lemma 3.1, this torsion subgroup is of the form $H \oplus H$ so we may assume the former holds. Decomposing $\tau H^2(Y)$ as a direct sum of cyclic groups of prime power order we see that it is

$$\mathbb{Z}/2^{t_1+1} \oplus \mathbb{Z}/2^{t_1+1} \oplus K \oplus K,$$

for some $K$ while $\operatorname{coker} Q_X$ is

$$\mathbb{Z}/2^{t_1} \oplus \mathbb{Z}/2^{t_1} \oplus K \oplus K.$$

Also $H_i = \operatorname{im} q_2 \circ \iota_3^i \circ \beta^i$ is a subgroup of $q_2(\mathbb{Z}/2^{t_1+1} \oplus K)$. Since this is a square root order subgroup of $\operatorname{coker} Q_X$, it follows that this cokernel is isomorphic to $H_i \oplus H_i$.

To see $H_1$ and $H_2$ have the required intersection, note that they are images of maps which factor through $\iota_3^1$ and $\iota_3^2$. The images of these maps have trivial intersection by Lemma 3.1. Since $q_2$ takes the quotient by a subgroup of order two, $H_1$ and $H_2$ have at most two points of intersection in $\dfrac{H^2(Y)}{\langle\delta(t)\rangle}$. □

## 3.3 Linear subsets

In this section, we will prove the following theorems.

**Theorem 3.16.** *Let $L = \#_{i=1}^h L(p_i, q_i)$. Then $L$ embeds smoothly in $S^4$ if and only if each $p_i$ is odd and there exists $Y$ such that $L \cong Y\# - Y$.*

**Theorem 3.17.** *Let $Y$ be a Seifert manifold with non-orientable base surface $F$. If $Y$ embeds smoothly in $S^4$ then the Seifert invariants of $Y$ occur in weak complementary pairs. In addition, whenever there are Seifert invariants $(a_i, b_i), (a_j, b_j)$ with $a_i, a_j$ both even, then $a_i = a_j$ and $b_i \in \{\pm b_j, \pm b'_j\}$.*

To do this, we describe the necessary combinatorics. Let $\mathbb{D}^n$ be the lattice $\mathbb{Z}^n = \langle e_1, \dots e_n \rangle$ with respect to the product given by $-Id$.

**Definition 3.18.** *A subset $S = \{v_i\}$ of $\mathbb{D}^n$ is called linear if*

$$v_i \cdot v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j \\ 0 \text{ or } 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1. \end{cases} \tag{3.1}$$

A weighted graph $\Gamma(S)$ can be associated to every linear subset $S$ as follows. For each element $v_i$ there is a vertex with weight $v_i \cdot v_i$ and there is an edge connecting the vertices corresponding to $v_i$ and $v_j$ if and only if $v_i \cdot v_j = 1$. We will use the same notation for both the vector $v_i$ and the corresponding vertex. Define $c(S)$ to be the number of connected components of the graph $\Gamma(S)$.

Let $Q_\Gamma = -A(S)A(S)^t$ be the incidence matrix of $\Gamma$.

Define

$$G(S) = \frac{\mathbb{Z}^n}{\operatorname{im} Q_\Gamma} \text{ and } H(S) = \frac{\mathbb{Z}^n}{\operatorname{span} S} \cong \frac{\operatorname{im} A(S)}{\operatorname{im} Q_\Gamma}.$$

**Definition 3.19.** *A linear subset $S$ is called a linear double subset if $G(S) \cong H(S) \oplus H(S)$.*

Linear subsets were studied extensively by Lisca [Lis07a], [Lis07b]. It will be useful to review some of these ideas.

A pair of vectors $v, v'$ are called linked if there is some unit basis vector $e_i$ in $(\mathbb{Z}^n, -Id)$ such that $v \cdot e_i$ and $v' \cdot e_i$ are both nonzero. A subset $S$ is called irreducible if for any pair of vectors $v, v' \in S$ there is a sequence $v = w_1, \dots, w_k = v'$ such that each $w_i$ is linked to $w_{i+1}$. In [Lis07b] irreducible linear subsets were called good.

**Lemma 3.20.** *Let $S$ be a linear subset. If $S$ is not irreducible then $S = \cup_i T_i$ where each $T_i$ is irreducible and consists of $n_i$ vectors which are supported on $n_i$ of the basis vectors $\{e_j\}$.*

*Proof.* This is proved on page 2162 of [Lis07b]. ☐

We now look at how to describe the groups $H(S)$ and $G(S)$ in terms of the decomposition into irreducible subsets.

For a linear subset $S$, the connected components of the graph $\Gamma(S)$ are all linear weighted trees. If $Q_i$ denotes the incidence matrix of the $i^{th}$ tree we can arrange that $Q_\Gamma$ is the diagonal block matrix $\mathrm{diag}(Q_1, \dots Q_h)$. If the subset $S$ gives matrix $A$ then this has the form $\mathrm{diag}(A_1, \dots A_k)$ where each $A_j$ comes from an irreducible subset $T_j$. The group $H(S)$ also splits up as a direct sum with summands of the form

$$H(T_i) = \frac{\mathrm{im}\, A_i}{\mathrm{im}\, Q_{i_1} \oplus \dots \oplus Q_{i_{c(T_i)}}}.$$

**Proposition 3.21.** *Let $S$ be a linear double subset and suppose $S$ decomposes as $S = \cup_{i=1}^k T_i$ where each $T_i$ is irreducible. Then $H(T_i)$ is a square root order direct summand of $G(T_i)$ for each $i$.*

*In addition, if $c(T_i) = 2$ then $T_i$ is also a linear double subset.*

*Proof.* For each $1 \le j \le k$, let $G' = \bigoplus_{i \ne j} G(T_i)$ and $H' = \bigoplus_{i \ne j} H(T_i)$. Consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H(T_j) & \xrightarrow{\iota} & G(T_j) & \longrightarrow & \dfrac{G(T_j)}{H(T_j)} & \longrightarrow & 0 \\
 & & \uparrow\downarrow & & \uparrow\downarrow & & & & \\
0 & \longrightarrow & H(T_j) \oplus H' & \overset{\leftarrow}{\longrightarrow} & G(T_j) \oplus G' & \overset{\leftarrow}{\longrightarrow} & H(T_j) \oplus H' & \longrightarrow & 0. \\
 & & \uparrow\downarrow & & \uparrow\downarrow & & & & \\
0 & \longrightarrow & H' & \longrightarrow & G' & \longrightarrow & \dfrac{G'}{H'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It follows from the description above that this diagram commutes. The rows and columns are exact, with the obvious inclusion and quotient maps, and the first two columns and second row split.

There is then a map $\rho$ from $G(T_j) \to H(T_j)$. It is not hard to check that this splits the first row as well.

Thus $G(T_j) \cong H(T_j) \oplus K_j$ for some $K_j$ with the same order as $H(T_j)$. If $c(T_j) = 2$ then $G_j$ can be written as a sum of two cyclic groups. Both $H(T_j)$ and $K_j$ must be cyclic groups. □

The following special case can be observed immediately.

**Corollary 3.22.** *If $S$ is a linear double subset then every irreducible $T_i$ has $c(T_i) \geq 2$.*

We review a few more important notions from [Lis07a], namely the quantity $I(S)$, contractions of subsets and bad components.

**Definition 3.23.** *Let $S = \{v_i\}_{i=1}^m$ be a subset of $\mathbb{D}^m$. Define*

$$I(S) = \sum_{i=1}^{m} -v_i \cdot v_i - 3.$$

Note that $I(S)$ can be computed from weights of the graph $\Gamma(S)$. We will also use $I(C)$ when $C$ is a connected component of the graph by just summing over vectors corresponding to vertices in this component.

**Definition 3.24.** *Let $S \subset \mathbb{D}^m$ be a subset $\{v_i\}$ for which $|v_i \cdot e_j| \leq 1$ for each $i, j$. If there are $j, s, t$ such that $|v_i \cdot e_j| = 1$ if and only if $i \in \{s, t\}$ and $v_t \cdot v_t < -2$ then the subset $S' = S \backslash \{v_s, v_t\} \cup v_t'$ of $\mathbb{D}^{n-1}$ considered as the span of $\{e_k\}_{k \neq j}$ and where $v_t'$ is obtained from $v_t$ by removing the $e_j$ component is said to be obtained via a contraction of $S$.*

*Conversely, $S$ is called an expansion of $S'$.*

In the particular case where $v_s$ is a leaf of the graph and $v_s \cdot v_s = -2$ we will say that $S$ is an expansion of $S'$ by a final $(-2)$ vector.

**Definition 3.25.** *Let $S'$ be a linear subset of $\mathbb{D}^m$. Suppose that the subset $\{v_{s-1}, v_s, v_{s+1}\}$ is a connected component $C'$ of $\Gamma(S')$ and that there are $i, j$ such that $v_{s-1}$ and $v_{s+1}$ are both of the form $\pm(e_i \pm e_j)$ and $v_s \cdot v_s < -2$.*

*Let $S$ be any subset which is obtained from $S'$ by a sequence of expansions by final $(-2)$ vectors which belong to the connected component $C$ of $\Gamma(S)$ corresponding to $C'$.*

*The component $C$ is called a bad component of $S$.*

*We then define $b(S)$ to be the number of bad components of $S$.*

Note that the conditions on $S'$ mean that, up to reordering or a change of sign, $v_{s-1} = e_i - e_j$, $v_s = e_j + \dots$ and $v_{s+1} = -e_i - e_j$. Since every other element of $S'$ has product zero with $v_{s-1}$ and $v_{s+1}$, none contains a nonzero multiple of $e_i$ or $e_j$.

We may form a new subset $S''$ of $\mathbb{D}^{m-2}$ from $S'$ by deleting the elements $v_{s-1}$ and $v_{s+1}$ from the subset and deleting the basis vectors $e_i$ and $e_j$.

Note that the bad component $C'$ of $S'$ is necessarily given by a chain of length three with weights $-2, -n-1, -2$ for some $n \geq 2$. The corresponding component $C''$ of $S''$ is simply an isolated vertex with weight $-n$. We will call $C''$ in $S''$ the reduced component corresponding to $C$ in $S$.

We summarise the relevant features of bad components below.

**Proposition 3.26.** *Let $S$ be a linear subset with a bad component $C$. Suppose $C = \{v_1, \dots v_s\}$ and $S \backslash C = \{w_1, \dots w_r\}$. Then, possibly after reordering $\{e_i\}$, $w_i \cdot e_j = 0$ for all $j < s$. Also, there is some $1 < t < s$ such that whenever $j \geq s$ and $v_i \cdot e_j \neq 0$ then $i = t$.*

*Furthermore, the 4-manifold defined by plumbing on according to the component $C$ has boundary $L(m^2n, mnk+1)$, where $-n$ is the weight on the reduced component corresponding to $C$ and $m, k$ are coprime integers with $m > k > 0$. In addition, $I(C) = n - 4$.*

*Proof.* In the case where $s = 3$, this description follows from the discussion above. For $s > 3$, it follows from the definition of expansion by a final $(-2)$ vector. The claim that $C$ represents $L(m^2n, mnk + 1)$ is the content of [Lis07b, Lemma 3.3]. It is apparent that $I(C)$ does not change under expansion by a final $(-2)$-vector. A simple calculation verifies the dependence on $n$. □

The key results concerning bad components are the following:

**Proposition 3.27.** *Let $S$ be a linear double subset with $c(S) = 2$. Then $S$ does not have a bad component.*

*Proof.* Suppose $S$ has a bad component so $G(S) \cong \mathbb{Z}/m^2n \oplus \mathbb{Z}/k$ for some $k$. This has square order so there is some $q$ so that $nk = q^2$. Every element of $H(S) \oplus H(S)$ has order dividing $mq$ and this implies that $k = mq = m^2n$.

Then we may assume that $G(S) = \left(\mathbb{Z}/m^2n\right)^2$ for some $n, m \geq 2$.

We will show that if $S$ has a bad component then every element of $H(S)$ has order dividing $mn$ and so $H(S) \oplus H(S)$ is not $\left(\mathbb{Z}/m^2n\right)^2$.

Letting $r, s$ be as in Proposition 3.26, $H(S)$ is the subgroup of $\frac{\mathbb{Z}^{s+r}}{\operatorname{im} Q_\Gamma}$ generated by the columns of $A(S)$. Our aim is to show that multiplying each column by $mn$ gives an element of $\operatorname{im} Q_\Gamma$.

The matrix $Q_\Gamma$ can be split up as $Q_1 \oplus Q_2$ where $Q_1$ is the $s \times s$ matrix corresponding to the bad component $C$ and $Q_2$ is the $r \times r$ matrix coming from the other component. We compare $S$ to other subsets with similar columns and incidence matrices.

Consider the first $s$ rows of $A(S)$. Proposition 3.26 tells us that, after suitable reordering, all but row $t$ has all its non-zero entries in the first $s-1$ columns. Therefore the last $r+1$ columns contain at most one non-zero entry which is in position $t$. As far as the order condition we are checking is concerned, it may be assumed that $n$ have entry $+1$ here and all others have entry zero. Now consider the subset $\overline{S}$ of $s+n-1$ vectors in $\mathbb{Z}^{s+n-1}$ where the first $s$ are the same as in $S$, except perhaps for the deletion of zero columns, and the last $n-1$ are given by $w_1 = e_s - e_{s+1}, \dots w_{n-1} = e_{s+n-1} - e_{s+n}$. Note that the matrix of this subset has the same first $s$ rows as $A(S)$. The graph of $\overline{S}$ consists of the bad component $C$ and a chain of $n-1$ vertices of weight $-2$. The incidence matrix for this graph is given by $\overline{Q} = Q_1 \oplus Q_3$ where $Q_3$ is the incidence matrix for the chain of $-2$'s. This presents $\mathbb{Z}/m^2n \oplus \mathbb{Z}/n$. The group $H(\overline{S})$ is of square root order so every element has order dividing $mn$. This shows that, for each of the columns with $s$ rows appearing in the upper part of $A(S)$, the vector given by multiplying by $mn$ is in the image of $Q_1$.

Now consider the last $r$ rows of $A(S)$. By Proposition 3.26, each of these has all the first $s$ entries zero. Let $S'$ be the subset obtained from $S$ by replacing the bad component $C$ with the corresponding reduced component. This has a matrix with $r+1$ rows and columns and the last $r$ rows differ from those of $S$ only by the removal of the columns containing only zeros. The graph of $S'$ consists of the component of $S$ corresponding to $S \backslash C$ and an isolated vertex with weight $-n$, so $G(S') = \mathbb{Z}/m^2n \oplus \mathbb{Z}/n$. Arguing as above, we see that the columns given by the last $r$ rows of each column of $A(S)$ gives an element of the image of $Q_2$ when multiplied by $mn$.

Thus, every column of $A(S)$ represents an element of order dividing $mn$ in $G(S)$, as claimed. □

The following technical result about subsets where every component is bad is also necessary. It will be convenient to introduce the following terminology. We call a subset $S$ of $m$ vectors in $\mathbb{D}^n$ square if $m = n$ and rectangular if $m = n+1$. Note that when $S$ is rectangular the matrix $Q_{\Gamma(S)} = -A(S)A(S)^t$ is singular.

**Proposition 3.28.** *If $S$ is a linear subset with $b(S) = c(S) = -I(S)$, $G(S)$ is not isomorphic to $H(S) \oplus H(S)$.*

*Proof.* Since every component of $S$ is bad the group $G(S)$ is a direct sum of cyclic groups of the form $\mathbb{Z}/m_i^2 n_i$ $(1 \leq i \leq c(S))$. The condition that $c(S) + I(S) = 0$ implies that

$$c(S) + \sum_{i=1}^{c(S)} (n_i - 4) = -3c(S) + \sum_{i=1}^{c(S)} n_i = 0. \tag{3.2}$$

By definition, each $n_i \geq 2$. By Proposition 3.26, there is a square linear subset $S'$ whose graph is given by $c(S)$ isolated vertices with weights $-n_i$. Suppose some $n_k = 2$. The vector $v_k$ in $S'$ with $v_k \cdot v_k = -2$ can be linked to other vectors $v_j$. Suppose that for each of these vectors $n_j \geq 3$. Then, by deleting $v_k$ and the columns on which it is supported, we get a rectangular subset with graph given by isolated vertices with weight $-n_i$ or $-n_j + 2$ with $n_j \geq 3$. This is not possible as the incidence matrix has non-zero determinant. A similar argument shows that $v_k$ must be linked to some $v_j$.

We now consider the possibility that some $n_j = 2$. In this case $v_k$ can only be linked to the corresponding vector $v_j$ in $S'$ so it follows that there is a decomposition of $S$ as $T \cup T'$ where $T$ consists of the bad components built from $v_k$ and $v_j$. It now follows from Propositions 3.21 and 3.27 that $S$ is not a linear double subset.

We now turn to the case where each $n_i$ is at least 3. Condition (3.2) then implies that $n_i = 3$ for each $i$.

Now, again, we can modify the subset $S$. For each bad component $C_i$, let $Q_i$ be the incidence matrix. Each $Q_i$ presents $\mathbb{Z}/3m_i^2$. The collection of rows of $A(S)$ corresponding to $Q_i$ is described by Proposition 3.26. In particular, we can obtain a subset $S'$ for the graph given by $C_i$ and a chain of 2 vertices of weight $-2$ as in the proof of Proposition 3.27 by extracting the rows corresponding to $C_i$ from $A(S)$, modifying the central row with square $-3$ so that every entry is zero or one, deleting any zero columns and adding a pair of new rows given by $w_1 = e_t - e_{t+1}, w_2 = e_{t+1} - e_{t+2}$.

Then $H(S')$ has order $3m_i$. Let $M$ be the least common multiple of $\{3m_i\}_{i=1}^{c(S)}$. It follows that $MH(S) = 0$.

Find a prime power $p^k$ and $i \in [1, c(S)]$ such that $p^k$ divides $m_i$ and $p^{k+1}$ does not divide any $m_j$. There is an element in $G(S)$ of order $3p^{2k}$. However, it is clear $3p^{2k}$ does not divide $M$ and this shows that there is no element of this order in $H(S) \oplus H(S)$. $\square$

We say that a pair of components $C_1, C_2$ of a weighted graph are complementary if the manifolds $Y_i$ bounding the 4-manifolds produced by plumbing according to $C_i$ are such that $Y_1 \cong -Y_2$.

**Proposition 3.29.** *Let $S$ be a linear subset such that*

$$c(S) + I(S) \leq 0 \text{ and } b(S) + I(S) < 0. \tag{3.3}$$

*If $S = \cup_i T_i$ where each $T_i$ is irreducible with $c(T_i) \geq 2$ and $b(T_i) = 0$ then the graph of $S$ consists of pairs of complementary components.*

*In addition, for each $T_i$, there are generators $t, s$ for $G(T_i)$ such that $H(T_i)$ is generated by $t + s$ or $t - s$.*

*Proof.* By [Lis07b, Proof of Lemma 5.5] there is at least one $T_i$ which satisfies (3.3). By [Lis07b, Proposition 4.10] this must have $c(T_i) = 2$.

It is shown in the proof of [Lis07b, Lemma 5.4] that since $T_i$ has no bad compoments, it is as described in [Lis07b, Lemma 4.7]. Thus plumbing on the graph of $T_i$ gives a manifold with boundary $L(p_i, q_i)\#L(p_i, p_i - q_i)$ for some $p_i, q_i$. This means that $c(T_i) + I(T_i) = 0$ by [Lis07a, Lemma 2.6]. We can apply the same argument to each irreducible subset $T_i$ since it follows that (3.3) must hold for each.

When $c(T) = 1$, a simple induction argument on the length $l$ of the chain shows that $G(T) = \mathbb{Z}^l / \operatorname{im} Q_{\Gamma(T)}$ is generated by $r = (1, 0, \ldots, 0)^t$. Similarly, when $c(T) = 2$ we easily find a pair of generators $t, s$ for $G(T) = \operatorname{coker} Q_1 \oplus Q_2$.

For every irreducible subset $T$ described by [Lis07b, Lemma 4.7] either $t + s$ or $t - s$ is the first column of $A(T)$ and thus represents an element of $H(T)$. It follows from comparing the orders that this generates the group. □

We may now prove Theorem 3.16 by combining the above results with some results of [Lis07b]. Theorem 3.17 is proved in precisely the same way.

*Proof of Theorems 3.16 and 3.17.* Suppose $Y$ embeds in $S^4$, and so, consequently, does $-Y$.

There is a negative definite 4-manifold with boundary $Y$, for either orientation. Applying Corollary 3.11 or 3.15 gives a linear double subset.

We may then choose an orientation. By [Lis07b, Lemma 5.3], we can assume that there is a linear double subset $S$ for which (3.3) holds.

Consider a decomposition of $S$ into irreducible components $S = \cup T_i$. By Corollary 3.22 each has $c(T_i) \geq 2$. Let $T$ be the union of all the $T_i$ which satisfy (3.3). Each of these must then have $c(T_i) = 2$ by [Lis07b, Proposition 4.10]. Since $S$ is a double subset it follows from Propositions 3.21 and 3.27 that $T$ has no bad components and we may apply Proposition 3.29.

Now consider $R = S \backslash T$. This is possibly not irreducible and has $c(R) + I(R) \leq 0$ since the corresponding quantity is at most zero for $S$ and is equal to zero for each $T_i$. In order to have no irreducible component satisfy (3.3), we must have $b(R) + I(R) \geq 0$. The fact that $b(R) \leq c(R)$ implies that $b(R) = c(R) = -I(R)$.

We require that $G(S) \cong H(S) \oplus H(S)$. Writing $S$ as the union of $R$ and $T$ gives $G(S) = G(T) \oplus G(R)$ and $H(S) = H(T) \oplus H(R)$.

It is clear that $G(T) \cong H(T) \oplus H(T)$. It follows that we must have $G(R) \cong H(R) \oplus H(R)$.

However, this contradicts the result of Proposition 3.28. We conclude that $R$ is empty. This proves that the Seifert invariants occur in weak complementary pairs as they are determined by $\Gamma$. Note that this graph does not distinguish between Seifert invariants of the form

$$\frac{a}{b} = [a_1, \ldots, a_n]^- \text{ and } \frac{a}{b'} = [a_n, \ldots, a_1]^-.$$

We now use the second linear double subset given by Corollary 3.11 or 3.15. Each subset $S_k$ $(k = 1, 2)$ is given by a union of irreducible subsets $T_{k,i}$, all of which satisfy (3.3). Since the graphs of $S_1$ and $S_2$ are identical, we will just write $G(S)$ instead of $G(S_i)$. For each ratio $a/b$ let $T_k^{a/b}$ be the set of irreducible $T_{k,i}$ whose graph represents $L(a, b)\#L(a, a - b)$. The union of these subsets gives a summand $(\mathbb{Z}/a)^{2l}$ of $G(S)$. By Proposition 3.29, this has generators $t_1, s_1, \ldots, t_l, s_l$ and we can arrange that $H(T_1^{a/b})$ is generated by $t_1 + s_1, \ldots, t_l + s_l$. There is a similar set of generators for $H(T_2^{a/b})$ given by $\sigma(t_1) \pm \sigma(s_1), \ldots, \sigma(t_l) \pm \sigma(s_l)$ where $\sigma$ is some permutation of $\{t_1, s_1, \ldots, t_n, s_n\}$.

When $a$ is even, $\frac{a}{2}\,(t_1 + s_1 + \ldots + t_l + s_l)$ is an element of $H(T_1^{a/b})$ and $H(T_2^{a/b})$. Theorem 3.16 now follows from Corollary 3.11. In the case of a non-orientable base orbifold, Corollary 3.15 implies that there can be at most one non-empty $T_i^{a/b}$ with $a$ even, completing the proof of Theorem 3.17.

□

**Remark 3.30.** The fact that any factor $L(p, q)$ in a connected sum of lens spaces embedding in $S^4$ has $p$ odd also follows from the linking form [KK80].

It is sometimes convenient to classify $S^1 \times S^2$ as a lens space since it also has a genus one Heegaard splitting.

**Corollary 3.31.** *Let $L = \#L(p_i, q_i)$ with $p_i > q_i > 0$ and suppose $L\#^n S^1 \times S^2$ embeds smoothly in $S^4$. Then $L$ also embeds smoothly.*

*Proof.* Replace the negative definite 4-manifold $X_L$ with boundary $L$ by $X_L \natural^n S^1 \times D^3$ and follow the proof of Theorem 3.16. □

A similar approach works for Seifert manifolds with $e = 0$.

**Theorem 3.32.** *Let $Y$ be a Seifert manifold with orientable base surface $F$ and $e(Y) = 0$. If $Y$ embeds smoothly in $S^4$ then the Seifert invariants of $Y$ occur in complementary pairs.*

*Proof.* Suppose $Y$ embeds smoothly in $S^4$.

By Corollary 3.14 we have a rectangular subset $S$. The graph $\Gamma(S)$ is star-shaped and has a semi-definite incidence matrix.

Deleting the vector in $S$ corresponding to the central vertex of $\Gamma$ gives a new subset $S'$. This subset is linear and has the additional property that there is a vector $v$ which links once to a leaf of each component of the graph of $S'$ and not to any other vector.

Note that we may choose either orientation for $Y$ and so can assume that $S'$ satisfies condition (3.3). We consider the irreducible components of $S'$. To apply Proposition 3.29 we need to show that every irreducible component $T$ has $c(T) \geq 2$ and $b(T) = 0$.

Suppose that $c(T) = 1$. Plumbing on the graph of $T$ gives the lens space $L(p, q)$ for some $p > q > 0$. There is an extra vector $v$ such that $T \cup \{v\}$ is a rectangular subset and has a linear graph obtained from that of $T$ by adding a vertex onto one end, with weight $t$. Since the subset is rectangular, it follows that the determinant of the incidence matrix of this graph must be zero. However we can easily see that the graph is negative definite, so we conclude $c(T) \geq 2$.

Now suppose that $T$ has a bad component $C$. By definition, this bad component can be built up from a linear chain of length three, with weights $-2, -n-1$ and $-2$ respectively. Let $C'$ be the component obtained from $C$ by deleting the vertex with weight $-n-1$. Suppose there is a new vertex $v$ which is only linked to one leaf of $C$ and consider the subset $T \cup \{v\}$. By Proposition 3.26 each of the $r$ components of $C'$ is supported on $r$ columns of the matrix for this subset. We may then get a rectangular subset $T'$ by deleting the other columns and every row corresponding to $T \backslash C'$. The resulting graph has two components and is obtained from $C$ by adding a new vertex of weight $t$ to one end and deleting the vertex of weight $-n-1$. Similar to above, the incidence matrix of this graph is negative definite and so we conclude that $b(T) = 0$.

It now follows from Proposition 3.29 that $Y$ has Seifert invariants occurring in (possibly weak) complementary pairs and, by [Lis07b, Proposition 4.10], that each irreducible $T_i$ has $c(T_i) = 2$. Adding a new row $v_i$ to each $T_i$ gives a linear graph, which is negative definite graph whenever $v_i \cdot v_i < -1$. This shows that each $v_i \cdot v_i = -1$ and the result now follows from the description of the irreducible subsets in Proposition 3.29 and [Lis07b, Lemma 4.7].

□

## 3.4 Obstructions from spin and spin$^c$ structures

The methods described in the previous sections are more difficult to implement and give weaker obstructions in the case of Seifert manifolds with orientable base surfaces and $e \neq 0$. We therefore look for additional obstructions. Since we have chosen to primarily consider the case of double branched covers of pretzel links, we will focus on applications to that case when convenient.

If $Y$ is a closed, oriented 3-manifold it admits spin and spin$^c$ structures. Suppose $Y$ embeds smoothly in $S^4$ and splits it as $S^4 = U \cup_Y -V$. The 4-manifolds $U$ and $V$ must have both spin$^c$ and spin structures. There are obstructions to $Y$ embedding in $S^4$ which can be found by looking at the spin and spin$^c$ structures on $Y$ which can be extended over either $U$ or $V$.

### 3.4.1 Spin$^c$ structures on rational homology spheres and the $d$ invariant

If a manifold $Y$ admits spin$^c$ structures then the set of spin$^c$ structures is a $H^2(Y;\mathbb{Z})$-torsor. Suppose that $Y$ is a rational homology sphere which embeds smoothly in $S^4$. This gives a pair of rational homology balls $U, V$ such that $S^4 = U \cup_Y -V$. The spin$^c$ structures on $Y$ which arise as the restrictions of spin$^c$ structures on $U$ correspond to the image of the inclusion map $H^2(U) \to H^2(Y)$. By Lemma 3.1, the inclusion maps induce an isomorphism

$$H^2(Y;\mathbb{Z}) \cong H^2(U;\mathbb{Z}) \oplus H^2(V;\mathbb{Z}).$$

Since these two summands are isomorphic, there are $k^2$ spin$^c$ structures on $Y$. At least $2k-1$ of these spin$^c$ structures extend over a rational ball since $k$ extend over each of the rational balls $U$ and $V$ and only one – the restriction of the unique spin$^c$ structure on $S^4$ – extends over both pieces.

The correction term, or $d$ invariant, from Heegaard-Floer theory is a $\mathbb{Q}$-valued invariant of a rational homology 3-sphere with a spin$^c$ structure, first introduced in [OzSz03a]. For our purposes, the relevant feature of this invariant is that whenever $(Y, \mathfrak{s})$ is a spin$^c$ 3-manifold and there is a rational ball $B$ bounding $Y$ with a spin$^c$ structure which restricts to $\mathfrak{s}$ on the boundary, then $d(Y, \mathfrak{s}) = 0$.

The $d$ invariant for a Seifert rational homology sphere can be determined using the associated star-shaped negative definite graph [OzSz03b] since it has at most one bad point.

It is described in [GJ11] how to relate this to the obstruction derived from Donaldson's theorem. This is used to obtain a stronger version of Theorem 3.8 in the case where $Y$ has the $\mathbb{Z}/2$-homology of $S^3$. We may restate [GJ11, Theorem 3.6] as follows.

**Theorem 3.33.** *Let $Y$ be a 3-manifold with $H_*(Y;\mathbb{Z}/2) \cong H_*(S^3;\mathbb{Z}/2)$ which smoothly bounds a rational ball. Suppose that $Y$ bounds a negative definite plumbing $X$ with at most two bad points. The vertices of this plumbing give a basis for $H_2(X)$ and we may then identify $H^2(Y;\mathbb{Z})$ with* $\operatorname{coker} Q_X$.

*Then there is a matrix $A$ such that $Q_X = -AA^t$ and every class of $\frac{\operatorname{im} A}{\operatorname{im} Q_X}$ contains a characteristic representative of the form $Ax$ for some $x \in \{\pm 1\}^n$.*

### 3.4.2 Spin structures, Furuta's theorem and the $\overline{\mu}$ invariant

In this section, we derive an embedding obstruction from Furuta's 10/8 theorem.

**Theorem 3.34** (Furuta [Fur01])**.** *Let $W$ be a closed, spin, smooth 4-manifold with an indefinite intersection form. Then*

$$4b_2(W) \geq 5|\sigma(W)| + 8.$$

Note that, by Donaldson's diagonalisation theorem, a closed, smooth, spin manifold $W$ can have a definite intersection form only if $b_2(W) = 0$.

In fact, we can use this to get an obstruction for a 3-manifold to be the boundary of a spin 4-manifold with 'small' homology. The essential idea here is simple. If $Y$ bounds a homologically small spin 4-manifold $V$, such as a rational ball, any other spin manifold $X$ with boundary $Y$ can be glued to $V$ to give a closed spin manifold as long as the spin structure restricts to $Y$ to give the same spin structure as the one obtained from $V$. The rank and signature of the intersection form of $X$ and $X \cup_Y -V$ will be similar and so a statement analogous to Furuta's theorem holds for spin 4-manifolds which share a boundary with a homologically small 4-manifold. In this section we make this claim precise. It turns out that, at least for Seifert manifolds, it is sensible to apply these results to the Neumann-Siebenmann $\overline{\mu}$-invariant.

If a manifold $Y$ admits a spin structure then the set of spin structures on $Y$ is a torsor for $H^1(Y;\mathbb{Z}/2)$. As with spin$^c$ structures, if $Y$ is a 3-manifold which embeds in $S^4$ there is an isomorphism induced by inclusion maps

$$H^1(Y;\mathbb{Z}/2) \cong H^1(U;\mathbb{Z}/2) \oplus H^1(V;\mathbb{Z}/2).$$

**Lemma 3.35.** *If $Y$ is the double branched cover of a $k$-component link $L$ then it has $2^{k-1}$ spin structures. If $Y$ embeds smoothly in $S^4$ then $b_1(Y)$ is even if and only if $k$ is odd. In particular, when $L$ is a pretzel link $b_1(Y)$ is zero when $k$ is odd and one when $k$ is even.*

*Proof.* By Corollary 3.2, the number of spin structures on a 3-manifold $Y$ embedding in $S^4$ is $2^{b_1(Y)}l^2$ for some integer $l$. This is a square precisely when the first Betti number is even. By [Tur88] there is a correspondence between quasiorientations of a link and spin structures on the double branched cover. For a $k$-component link there are $2^{k-1}$ spin structures on the double branched cover and this is a square precisely when $k$ is odd.

When $L$ is a pretzel link, $Y$ is a Seifert manifold with base $S^2$ and it follows from, for example, [Hil09, Theorem 3.1], that $b_1(Y) \leq 1$.

□

Let $Y$ be given as the boundary of a 2-handlebody $X$ represented by a framed link in $S^3$.

**Definition 3.36.** *A sublink $L'$ of a framed link $L$ is called characteristic if for every component $K$ of $L$ the total linking number $\operatorname{lk}(K, L')$ is congruent modulo 2 to the framing on $K$.*

Spin structures on $Y$ correspond bijectively to characteristic sublinks of the diagram for $X$ (see [GS99, Proposition 5.7.11] for example). The characteristic sublink of a spin structure $\mathfrak{s}$ represents an obstruction to extending $\mathfrak{s}$ over the 2-handlebody. If the characteristic sublink is empty, the 2-handlebody has a unique spin structure which restricts to $\mathfrak{s}$ on the boundary.

Kaplan [Kap79] gives an algorithm which produces a spin 2-handlebody extending a given spin structure on any 3-manifold. The algorithm uses handle-slides and blow-ups to remove the characteristic sublink. We briefly recall the effects of these moves on characteristic sublinks. (See [GS99, Section 5.7] for a more detailed discussion.) If we slide one component of a characteristic sublink over another the characteristic sublink in the new diagram simply contains the new curve, and so has one fewer component. The new curve added in a blow-up is included in a characteristic sublink if and only if it has an even linking number with the sublink. If we blow down a component in a characteristic sublink then the corresponding characteristic sublink in the resulting diagram consists of the other curves in the original.

Suppose that $X$ is given by plumbing on a tree $\Gamma$. The spin structures on the boundary of $X$ now correspond bijectively to subsets of the vertex set of $\Gamma$ which are characteristic for the incidence matrix of $\Gamma$. Such sets, or equivalently the classes they represent in $H_2(X;\mathbb{Z}/2)$, are called (homology) Wu sets and are always isolated.

**Definition 3.37.** *Let $X$ be a plumbing according to a weighted tree. The Neumann-Siebenmann $\overline{\mu}$ invariant of $Y = \partial X$ with spin structure $\mathfrak{s}$ corresponding to a Wu set $w$ is defined as $\overline{\mu}(Y,\mathfrak{s}) = \sigma(X) - w \cdot w$.*

It is shown in [Neu80] that this only depends on $(Y,\mathfrak{s})$ and not on the 4-manifold $X$ used in the construction. It is apparent that this is a lift of the Rochlin invariant.

We would like to consider the $\overline{\mu}-$invariant for Seifert manifolds with spin structures which extend over 4-manifolds with simple rational homology. The key result is Furuta's 10/8 theorem. For Seifert rational homology spheres the $\overline{\mu}$-invariant is known to be a spin rational homology cobordism invariant [Ue05] (see also [Sav02] for integer homology spheres), which is proved using a V-manifold version of the 10/8 theorem [FF00].

Here, we will give an alternative argument which is applicable for the cases we are most interested in, including some with positive first Betti number. Our approach is similar to [BL02], which derives a knot sliceness obstruction from Furuta's theorem.

**Lemma 3.38.** *Let $(Y,\mathfrak{s})$ be a 3-manifold with a chosen spin structure. Suppose that $(X,\mathfrak{s}_X)$ is a spin 2-handlebody and $(V,\mathfrak{s}_V)$ is a spin manifold with $b_3(V) = 0$ such that $\partial(X,\mathfrak{s}_X) = \partial(V,\mathfrak{s}_V) = (Y,\mathfrak{s})$.*

*Then $W = X \cup_Y -V$ is spin with signature $\sigma(W) = \sigma(X) + \sigma(V)$ and $b_2(W) = b_2(X) + \chi(V) - 1$.*

*Proof.* The fact that $W$ is spin follows since the spin structures on $X$ and $V$ agree on the boundary.

We may compute the signature and second Betti number as in Proposition 3.3: It is easy to see that $\chi(W) = \chi(X) + \chi(V) = 1 + b_2(X) + \chi(V)$. Since $H_1(W, X; \mathbb{Q}) \cong H_1(V, Y; \mathbb{Q}) = 0$ it follows from the exact sequence for the pair $(W, X)$ that $b_1(W) = 0$. The result now follows from the calculation of the Euler characteristic and Novikov additivity. □

To get an obstruction to a 3-manifold $Y$ with $b_1(Y) \leq 1$ embedding in $S^4$, we consider the case where $V$ is one of the spin pieces obtained from the splitting induced by an embedding.

**Corollary 3.39.** *Let $(Y, \mathfrak{s})$ be a spin 3-manifold and let $(V, \mathfrak{s}_V)$ be a spin manifold and $(X, \mathfrak{s}_X)$ be a spin 2-handlebody with common boundary $(Y, \mathfrak{s})$.*

1. *If $V$ is a rational ball then either $X = D^4$ or*
$$4b_2(X) \geq 5|\sigma(X)| + 8;$$
2. *If $H_*(V; \mathbb{Q}) = H_*(S^1; \mathbb{Q})$ then either $b_2(X) = 1$ or*
$$4b_2(X) \geq 5|\sigma(X)| + 12;$$
3. *If $H_*(V; \mathbb{Q}) = H_*(S^2; \mathbb{Q})$ then*
$$4b_2(X) \geq 5|\sigma(X) + \sigma(V)| + 4.$$

*Proof.* By Donaldson's theorem, the closed, spin manifold $W = X \cup_Y -V$ is definite only if $b_2(W) = 0$. Otherwise, we apply Furuta's theorem and Lemma 3.38 to $W$. □

We now construct spin 4-manifolds bounding double branched covers of pretzel links.

**Proposition 3.40.** *Let $Y$ be the double branched cover of a 3 or 4-stranded pretzel link and let $\mathfrak{s}$ be a spin structure on $Y$. Then there is a spin 2-handlebody $(X, \mathfrak{s}_X)$ with spin boundary $(Y, \mathfrak{s})$ with signature $\sigma(X) = \overline{\mu}(Y, \mathfrak{s})$ and $0 \leq b_2(X) - |\sigma(X)| \leq 4$.*

*Proof.* Let $X'$ be one of the 2-handlebodies shown in Figure 3.1. The boundary is the same as the 2-handlebodies pictured in Figure 1.14 – we can slide over the component with framing $a$ and then exchange the 0 framed unknot for a 1-handle and cancel.

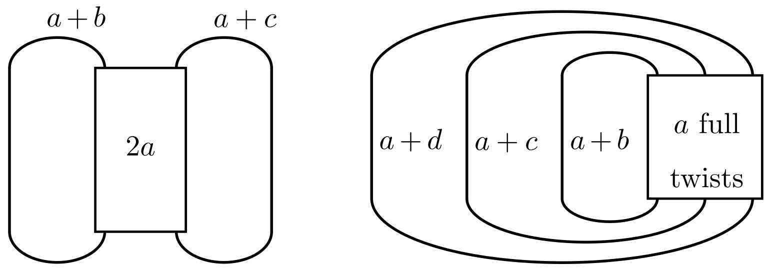



Figure 3.1: $X'$ **for** $n = 3, 4$**.**

Every sublink of $X'$ is potentially characteristic, depending on $a, b, c$ and $d$. For each spin structure $\mathfrak{s}$ on $\partial X'$ we can arrange by handleslides that the characteristic sublink is an unknot as follows. If the sublink containing the two components of framings $a + b$ and $a + c$ is characteristic we can slide one over the other to get a single unknotted component with framing $b + c$. In the 4-strand case, there may be a characteristic sublink with three components. If we perform the handle slide above, the resulting picture has a characteristic unlink. It is then obvious that we can slide one component over the other.

This gives a new diagram for $X'$ in which the characteristic sublink is an unknot with framing $n$. The $\overline{\mu}$ invariant of $(Y, \mathfrak{s})$ is $\sigma(X') - n$. This can easily be verified using the above description of the handle moves needed to convert the plumbing tree to $X'$.

By reversing the orientation of $X'$ if necessary, we may assume $\sigma(X') \geq 0$. Note that since $X'$ has only a small number of handles, $\sigma(X') \leq 3$. We now consider various cases depending on the sign of $n$.

If $n = 0$ then we can remove the characteristic sublink by blowing up a $+1$ meridian of it and then blowing down the resulting $+1$ framed curve. This gives an $X$ with signature $\overline{\mu}(Y, \mathfrak{s})$ and $b_2(X) = b_2(X') \leq 3$.

If $n < 0$ then the characteristic sublink can be removed by blowing up $|n| - 1$ meridians with framing $+1$ and then blowing down the resulting $-1$ curve. This produces a spin manifold $X$ with $\sigma(X) = \sigma(X') - n$ and $b_2(X) = b_2(X') + |n| - 2$.

By the assumptions on $X'$ and $n$, $\sigma(X) > 0$ so

$$\begin{aligned} b_2(X) - |\sigma(X)| &= b_2(X') - n - 2 - \sigma(X') + n \\ &= b_2(X') - \sigma(X') - 2. \end{aligned}$$

This is at most 1.

If $n > 0$, the characteristic sublink can be removed by blowing up a $-1$-framed meridian of the characteristic link $n-1$ times and blowing down a $+1$ curve. This gives a spin manifold $X$ with $\sigma(X) = \sigma(X') - n$ and $b_2(X) = b_2(X') + n - 2$.

If $\sigma(X) \geq 0$ then necessarily $n \leq 3$. Then $b_2(X) \leq b_2(X') + 1 \leq 4$. Alternatively, if $\sigma(X) < 0$ then

$$\begin{aligned} b_2(X) - |\sigma(X)| &= b_2(X') + n - 2 + \sigma(X') - n \\ &= b_2(X') + \sigma(X') - 2. \end{aligned}$$

This is, again, at most 4.

□

We can apply Corollary 3.39 to produce the following conclusions.

**Corollary 3.41.** *Let $Y$ be the double branched cover of a 3 or 4 stranded pretzel link with $k$ components. If $Y$ embeds in $S^4$ then the Neumann-Siebenmann $\overline{\mu}$ invariant vanishes for at least $2^{\frac{k+1}{2}} - 1$ spin structures on $Y$ if $k$ is odd and at least $3(2^{\frac{k-2}{2}}) - 1$ if $k$ is even.*

*Proof.* Since $Y$ embeds smoothly in $S^4$ we can write $S^4 = U \cup_Y -V$. Since $Y$ is the double branched cover of a pretzel link $b_1(Y) \leq 1$. Lemma 3.1 implies that for both $U$ and $V$ the sum of the first and second Betti number is at most one and the third Betti number is zero.

For every spin structure $\mathfrak{s}$ extending over either $U$ or $V$ we apply Corollary 3.39 to the 2-handlebody $X$ given by Proposition 3.40. This shows that

$$4b_2(X) \geq 5|\overline{\mu}(Y, \mathfrak{s})| + 4.$$

Since $b_2(X) \leq |\overline{\mu}(Y, \mathfrak{s})| + 4$ we see that $|\overline{\mu}(Y, \mathfrak{s})| \leq 12$.

Since $U$ and $V$ both have signature zero, it follows from Rochlin's theorem that the $\overline{\mu}$ invariant vanishes for every spin structure extending over $U$ or $V$.

The proof of Lemma 3.35 shows that the total number of spin structures on $Y$ is $2^{b_1(Y)}l^2$ where $2^{b_1(Y)}l$ spin structures extend over $U$ and $l$ extend over $V$. Exactly one extends over both to give the unique spin structure on $S^4$. The result now follows since $b_1(Y)$ is determined by the parity of $k$. □

## 3.5 Double branched covers of pretzel links

Recall that $Y(a, b, c)$ and $Y(a, b, c, d)$ denote the double branched covers of $P(a, b, c)$ and $P(a, b, c, d)$ respectively. The aim of this section is to prove the following result.

**Theorem 3.42.** *Let $Y$ be of the form $Y(a,b,c)$ or $Y(a,b,c,d)$ where $a,b,c \in \mathbb{Z}\backslash\{-1,0,1\}$ and $d \in \mathbb{Z}\backslash\{0\}$. If $Y$ embeds smoothly in $S^4$ then it is (possibly orientation-reversing) diffeomorphic to one of the following*

- $Y(a,-a,a)$;
- $Y(a,-a,a,-a)$;
- $Y(a,-a,b,-b)$ *with $b$ odd;*
- $Y(a\pm 1,-a,a,-a)$;
- $Y(2\lambda-1,-2\lambda-1,-2\lambda^2)$.

*In addition, all but the last of these do embed smoothly in $S^4$.*

The proof of Theorem 3.42 will use a combination of the obstructions from Sections 3.2 and 3.4. All of the positive embedding results are demonstrated in Chapter 2. This section will complete the proof by outlining the necessary obstructions.

It will be convenient to use Corollary 3.41 as our principal obstruction. Accordingly, we consider cases with different numbers of spin structures separately. By Lemma 3.35 this is equivalent to splitting up into cases according to the number of link components.

We first consider the cases with first Betti number one. Note that these fall under the hypothesis of Theorem 3.32 and so every example of this type which embeds smoothly in $S^4$ is of the form $Y(a,-a,b,-b)$.

**Proposition 3.43.** *Suppose that $a > b > 0$ and that $a$ and $b$ are both even. Then $Y = Y(a,-a,b,-b)$ does not embed smoothly in $S^4$.*

*Proof.* An easy calculation using the plumbing in Figure 1.14 shows that $Y$ has eight spin structures and that only four have vanishing $\overline{\mu}$ invariant. The others are $\pm(a\pm b)$. Corollary 3.41 shows that these do not embed smoothly in $S^4$. □

**Remark 3.44.** This demonstrates that Theorem 3.32 does not give a complete obstruction.

We now consider the double branched covers of links with odd numbers of components.

### 3.5.1 Double branched covers of knots

Due to interest in the question of knot sliceness, there are previous results we may appeal to. In particular, for pretzel knots, the possible form of subsets appearing in Theorem 3.8 have been computed [GJ11] [Lec10]. The $\overline{\mu}$ invariant is useful as an obstruction to a knot being slice since any 4-manifold with the $\mathbb{Z}/2$ homology of $D^4$ is necessarily spin. Indeed, for Montesinos knots the $\overline{\mu}$ invariant of the double branched cover agrees with the knot signature [Sav00] and the resulting obstruction is incorporated into the results of [GJ11] and [Lec10].

To begin with, we consider the double branched covers of 3-stranded pretzel knots. There are two cases to consider. We assume that $Y(a,b,c)$ has a positive generalised Euler characteristic and consider how many of $a,b,c$ are positive.

**Proposition 3.45.** *Let $Y(a,b,c)$ be the double branched cover of a knot with $a,b>1$ and $e(Y)>0$. Then if $Y$ embeds smoothly in $S^4$, $c<0$ and $Y$ is diffeomorphic to $Y(a,-a,a)$.*

*Proof.* First, note that if $c$ is also positive it is impossible to have a vanishing $\overline{\mu}$ invariant. Since $e(Y)>0$ we can use the top diagram in Figure 1.14 to see that, for a Wu set $w$, the $\overline{\mu}$ invariant is given by $2-w\cdot w$. If $c>0$ and $a,b>1$, it is not possible to find $w$ with $w\cdot w=2$.

The case where $Y(a,b,c)$ is a $\mathbb{Z}/2$ homology sphere with $a,b>0$ and $c<0$ is by Greene and Jabuka [GJ11]. Note that while they only explicitly consider the case where $a,b,c$ are odd, this is only important in their calculation of the knot signature and has no effect on their arguments using Donaldson's diagonalisation theorem or the $d$ invariant. Their Proposition 3.1 determines the possible subsets in this case to be uniquely determined up to a choice of a parameter $\lambda$ such that $-c=\lambda^2a+(\lambda+1)^2b$.

Greene and Jabuka use the $d$ invariant, in the way described in Theorem 3.33, to show that this $\lambda$ must be either $-1$ or $0$ if $Y$ is the boundary of a rational ball. This shows that $-c=a$ or $-c=b$. Note that $Y(a,b,-a)$ has first homology of order $a^2$ so it can only be a homology sphere if it is $S^3$. Otherwise, we may apply Corollary 3.11 to show that there must be a second subset. This means that both $\lambda=0$ and $\lambda=-1$ must be valid. It follows that $a=b=-c$. □

Now we consider the case where $Y(a,b,c)$ has just $a$ positive.

**Proposition 3.46.** *Let $Y=Y(a,b,c)$ be the double branched cover of a knot with $a>1$,*

*$b, c < -1$ and $e(Y) > 0$. Then if $Y$ embeds smoothly in $S^4$ then it is a homology sphere of the form $Y(2\lambda - 1, -2\lambda - 1, -2\lambda^2)$.*

*Proof.* Since $e(Y) > 0$, for a Wu set $w$, we can use the top diagram in Figure 1.14 to calculate the $\overline{\mu}$ invariant as $-2 - w \cdot w$. If $a, b$ and $c$ are all odd then $w \cdot w$ is zero. Therefore, we must have one of these numbers even. Since $b, c < -1$, the curve with framing $a$ must be in the Wu set. Up to relabeling $b$ and $c$, we conclude that $a + b = w \cdot w = -2$ and $c$ is even.

Consider the 4-manifold $X'$ with boundary $Y$ shown in Figure 3.2, where $2a$ refers to the number of crossings.

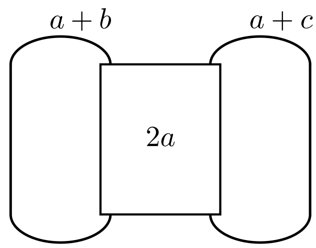



Figure 3.2: $X'$.

The intersection form of $X'$ has determinant $ab + ac + bc > 0$ so it is definite. Since $a + b = -2$ it must be negative definite.

The possible subsets we get from applying Corollary 3.11 give matrices of the form

$$A_i = \begin{pmatrix} 1 & 1 \\ \rho & \lambda \end{pmatrix}.$$

Up to a change of basis of the columns space this is unique. This means that there is only one subset. This provides an obstruction unless $Y$ is a homology sphere as noted in Remark 3.12. In this case we require that $\det A = \lambda - \rho = \pm 1$. Up to relabelling we can assume $\rho = \lambda - 1$.

It then follows that $a = 2\lambda - 1$, $b = -2\lambda - 1$ and $c = -2\lambda^2$. □

We now consider the double branched covers of 4-strand pretzel knots.

**Proposition 3.47.** *Let $Y = Y(a, b, c, d)$ with $a, b, c, d \in \mathbb{Z}\backslash\{0\}$ be a $\mathbb{Z}/2$ homology sphere which embeds smoothly in $S^4$. Then $Y$ embeds smoothly in $S^4$ if and only if it is diffeomorphic to $Y(a \pm 1, -a, a, -a)$.*

*Proof.* Suppose $Y$ embeds smoothly in $S^4$. We consider the condition imposed by Corollary 3.11.

In [Lec10, Lemma V.6] the subset obtained by viewing the standard negative definite plumbing as a submanifold of a closed definite manifold is uniquely determined and, in conjunction with the $\overline{\mu}$ invariant, is used to show that if $Y$ bounds a rational ball it is either $Y(-a, -b-1, a, b)$ with $a, b < -1$ or, if $a = 1$, has the form $Y(1, -2, b, -b) \cong Y(2, b, -b)$. The latter is considered above and does not embed smoothly in $S^4$.

The subset $S$ for $Y(-a, -b-1, a, b)$ is described explicitly by [Lec10, Figure V.5] and is obtained by adding a new column with a single non-zero entry to the matrix for the essentially unique rectangular subset for $Y(a, -a, b, -b)$.

On inspection we see that in order to get a second subset, which differs as specified by Corollary 3.11, we must have $a = b$. □

### 3.5.2 Double branched covers of 3-component links

Finally we consider double branched covers of pretzel links with three components. By Lemma 3.35 the double branched covers have four spin structures and, if they embed in $S^4$, are rational homology spheres.

We first consider the following special case, where Corollary 3.41 is not sufficient.

**Proposition 3.48.** *Let $a$ be odd and $b$ even. If $Y(a, b, b, b)$ embeds smoothly in $S^4$ and has $e(Y) > 0$ then it is diffeomorphic to $Y(2, -2, 2)$.*

*Proof.* In order to find a subset, $b$ must be negative or 2. We can see this by a simple extension of the proof of [Lec10, Lemma V.5], where we drop the assumption that $Y$ is a $\mathbb{Z}/2$ homology sphere – we attempt to construct a subset and compare the number of columns required to the number of vertices in the graph. The $\overline{\mu}$ invariants for $Y(a, 2, 2, 2)$ can easily be calculated and three are $\operatorname{sign} a - a$. The manifold $Y(-1, 2, 2, 2) \cong Y(2, -2, 2)$ embeds in $S^4$ but $Y(1, 2, 2, 2)$ does not as it has first homology of non-square order 20.

In the case where $b < 0$, the generalised Euler invariant implies that $a > 0$. Calculating the $\overline{\mu}$ invariants shows that $a = -b - 3$. The condition that $a > 0$ means that $b < -3$.

We can now express the generalised Euler characteristic as

$$\frac{1}{-b-3} + \frac{3}{b} = \frac{-2b-9}{-b^2-3b} > 0.$$

Since the denominator in this fraction is $ab < 0$, this shows that $b \geq -4$.

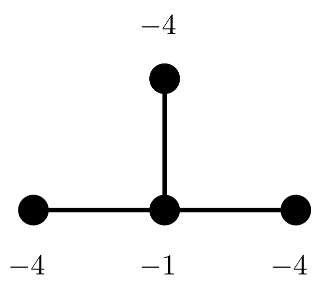



Figure 3.3: **Standard negative definite plumbing for** $Y(1,-4,-,4,-4)$**.**

To show that $Y(1,-4,-4,-4)$ does not embed in $S^4$, we use Corollary 3.11.

For the standard definite plumbing, shown in Figure 3.3 a simple computation shows that the matrix $A(S)$ is uniquely determined up to reordering or changing the signs of the columns and must be

$$A(S) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

. □

Finally, we consider the last remaining case needed to prove Theorem 3.42

**Proposition 3.49.** *Let $Y$ be of the form $Y(a,b,c)$ or $Y(a,b,c,d)$ where $a,b,c,d \in \mathbb{Z}\backslash\{0\}$. Suppose that $Y$ has four spin structures. Then $Y$ embeds smoothly in $S^4$ if and only if it is diffeomorphic to either $Y(a,-a,a)$ or $Y(a\pm 1,-a,a,-a)$.*

*Proof.* We first consider $Y = Y(a,b,c)$. This has four spin structures only when $a,b$ and $c$ are even. Let $\tau$ be the signature of the 4-manifold given by the first diagram in Figure 1.14. The four $\overline{\mu}$ invariants of $Y$ are $\tau, \tau-a-b, \tau-a-c$ and $\tau-b-c$. Three are zero which implies that either $\tau = 0$ and, up to reordering, $a=b=-c$ or $a=b=c$. In the latter case $\tau = \pm 2$ and so $a=b=c=\pm 1$. This does not embed in $S^4$ as it is either the lens space $L(3,1)$ or $L(3,2)$.

Next, consider $Y = Y(a,b,c,d)$. This has four spin structures if exactly one is odd, which can be assumed to be $a$. Define $\tau$, similar to the above, using the second picture in Figure 1.14. The $\overline{\mu}$ invariants are $\tau-a-b, \tau-a-c, \tau-a-d$ and $\tau-a-b-c-d$.

We again require that three are zero. If the the last of these is not, we may apply Proposition 3.48. Otherwise, up to relabeling, $b=c=-d$. It follows easily, by considering the value of $\tau$ for either sign of $b$, that $a = -b \pm 1$. □

## 3.6 Embedding in a homology $S^4$

The obstructions considered thus far give obstructions to embedding a 3-manifold smoothly in any homology 4-sphere.

It is somewhat surprising that there are no known examples of 3-manifolds which embed smoothly in a homology 4-sphere but not in $S^4$. Some 3-manifolds are known to embed in homology spheres but are not known to embed in $S^4$ (see [BB12]; for example whenever $M$ is a homology 3-sphere, $M\# -M$ embeds in a homology $S^4$).

It seems reasonable to expect that more 3-manifolds can be embedded in homology 4-spheres than in $S^4$. This section describes an obstruction to embedding smoothly in $S^4$ which appears to rely on the fundamental group. It *may* therefore be able to detect examples which do not embed in $S^4$ but do embed in a homology $S^4$.

**Lemma 3.50.** *Let $U$ be a 4-dimensional connected submanifold of $S^4$ with boundary $Y$ and let $X$ be a simply connected 4-manifold with a boundary component homeomorphic to $Y$.*

*Then $U \cup_Y -X$ is simply connected.*

*Proof.* Let $V$ be the complement of $U$ in $S^4$. Take group presentations $\langle G_U; R_U\rangle$ and $\langle G_V; R_V\rangle$ for $\pi_1(U)$ and $\pi_1(V)$ respectively. Choose a generating set $G_Y$ for $\pi_1(Y)$.

We let $\iota_A$ denote the map on the fundamental group induced by the inclusion of $Y$ into $A$ for $A = X, U, V$.

Applying Seifert-van Kampen gives presentations

$$\langle G_U, G_V; R_U, R_V, \{\iota_U(\gamma)\iota_V(\gamma)^{-1}\}_{\gamma\in G_y}\rangle$$

for $\pi_1(S^4)$ and

$$\langle G_U; R_U, \{\iota_U(\gamma)\}_{\gamma\in G_y}\rangle$$

for $\pi_1(U \cup_Y -X)$.

Mapping generators $G_U$ by the identity map and $G_V$ by the trivial map induces a surjective homomorphism from $\pi_1(S^4)$ to $\pi_1(U \cup_Y -X)$. Since $S^4$ is simply connected, the result follows. □

We use this lemma in conjunction with the following corollary to Taubes' theorem on end-periodic manifolds, attributed to Akbulut.

**Proposition 3.51.** *[ [Tau87, Proposition 1.7]]Let $\Sigma$ be a homology 3-sphere which bounds a smooth 4-manifold with nonstandard, definite intersection form and with $\pi_1 = 1$. Then $\Sigma\# - \Sigma$ does not bound a definite 4-manifold with $\pi_1 = 1$.*

**Corollary 3.52.** *Let $\Sigma$ be as in the above Proposition. If $W : \Sigma\# - \Sigma \to Y$ is a smooth, definite and simply connected cobordism to a rational homology 3-sphere $Y$, then $Y$ does not embed smoothly in $S^4$.*

*Proof.* If $Y$ had a smooth embedding, it would bound a rational homology ball which was a submanifold of $S^4$. Gluing this to the cobordism would, by Lemma 3.50, give a manifold contradicting Proposition 3.51. □

**Remark 3.53.** The requirement that $Y$ be a rational homology sphere may be relaxed if there are sufficient conditions on the cobordism. The key is that a definite manifold can be obtained.

In practice, it is not too difficult to find surgery diagrams of 3-manifolds which are obstructed by this. Starting from an integer surgery diagram for a suitable choice of $\Sigma$, such as the Poincaré homology sphere, we may add 2-handles to kill the fundamental group. The framings of these can be chosen so that the resulting cobordism is definite.

However, we know relatively little about these 3-manifolds. It is not clear that we can find one which (we know) embeds in a homology sphere.

**Remark 3.54.** The idea here is somewhat similar to [Auc93], which uses Proposition 3.51 to find a irreducible homology sphere which cannot be obtained by Dehn surgery on a knot in $S^3$.

# Chapter 4

# Concordance of links

A useful tool for examining the knot concordance group is the double branched cover. This gives a homomorphism from $\mathcal{C}$ to the rational cobordism group of rational homology spheres. A useful way to think of the set of oriented knots is as a commutative monoid with involution. The operation here is connected sum and the involution is $K \mapsto -K$ where $-K$ is the mirror of $K$ with the orientation reversed.

Connected sums of links are not well-defined in general so we only consider links with a marked oriented component and define the connected sum of two such links by using the marked components. If the entire link is oriented we call it a *marked oriented link* while if only the marked component is oriented we will call it *partly oriented*. Figure 4.1 shows partly oriented and marked oriented links.

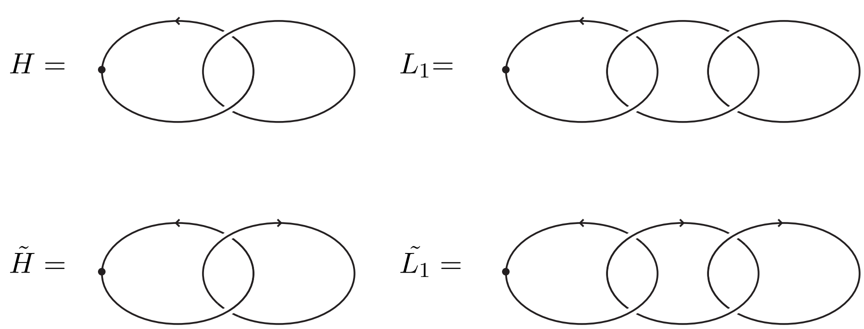



Figure 4.1: **Some partly oriented links $H$ and $L_1$ and marked oriented versions $\tilde{H}$ and $\tilde{L_1}$.**

The monoid of oriented knots is naturally included in the monoids of partly oriented links or marked oriented links and each marked oriented link gives rise to a partly oriented link by ignoring the orientations on each non-marked component. Each of these monoids has an involution $L \mapsto -L$ given by taking the mirror with all orientations reversed.

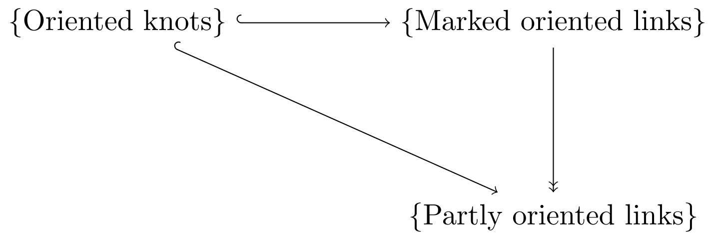



Figure 4.2: **Relationship between monoids with involution.**

The smooth knot concordance group $\mathcal{C}$ is obtained from this monoid of knots by taking the quotient under the equivalence relation that states $K_0$ is equivalent to $K_1$ if $-K_0\#K_1$ is smoothly slice. A variant – the topological concordance group $\mathcal{C}_{TOP}$ – is obtained by taking the quotient by topologically slice knots. In either case the class of $-K$ gives the inverse of $K$.

The (smooth) rational homology cobordism group of rational homology spheres $\Theta^3_{\mathbb{Q}}$ is the set of equivalence classes of closed oriented 3-manifolds with $b_1 = 0$ up to the relation $Y_0 \sim Y_1$ if and only if there is a (smooth) cobordism $W : Y_0 \to Y_1$ with $H_*(W;\mathbb{Q}) = H_*(S^3 \times I;\mathbb{Q})$. This is a group with operation given by connected sum. The equivalence relation can also be stated as $Y_0 \sim Y_1$ if and only if $-Y_0\#Y_1$ bounds a smooth rational homology ball. A key property of $\mathcal{C}$ is that the double branched cover gives a well-defined homomorphism $\Sigma_2 : \mathcal{C} \to \Theta^3_{\mathbb{Q}}$.

Our aim is therefore to find concordance groups of links so that the commuting diagram in Figure 4.2 passes to a quotient level and so that the homomorphism $\Sigma_2$ lifts to, if not the whole link concordance group, a large subgroup containing $\mathcal{C}$.

The key result from this point of view is then the following.

**Proposition 4.1.** *Let $F$ be a locally flat properly embedded surface in $D^4$ with no closed components and Euler characteristic $n$. Suppose that the boundary of $F$ is a link with non-zero determinant. Then the double cover of $D^4$ branched along $F$ has $b_1 = b_3 = 0$ and $b_2 = 1 - n$.*

The surface here does not have to be connected or oriented. In the case where $F$ is a ribbon surface and $n = 1$ this is proved in [Lis07b, Lemma 3.6]. For smoothly embedded

$F$ one could appeal to [LW95].

*Proof.* The general strategy of the proof follows that of [KT76, Theorem 3.6]. Let $N = \Sigma_2(D^4, F)$ be the double cover of $D^4$ branched along $F$. We will construct $N$ by taking a double cover of $D^4 \setminus \nu F$, using a Gysin sequence to compute the homology, before regluing a copy of $\nu F$. We use $\mathbb{Z}/2$ coefficients throughout. The pair $(D^4, S^3)$ can be decomposed as $(D^4 \setminus F \cup \nu F, S^3 \setminus L \cup L \times D^2)$.

We use the Mayer-Vietoris sequence to find

$$H^1(D^4 \setminus F, S^3 \setminus L) \oplus H^1(\nu F, L \times D^2) \cong H^1(\partial \nu F \setminus \nu L, L \times S^1).$$

The intersection piece $A = \partial \nu F \setminus \nu L$ is an $S^1$-bundle over $F$ so we can use the relative Gysin sequence [AGP02, Theorem 11.7.36] to compare $H^1(\nu F, L \times D^2)$ to $H^1(A, L \times S^1)$. We get an exact sequence

$$0 \to H^1(F, L) \to H^1(A, S^1 \times L) \to H^0(F, L).$$

Since $F$ has no closed components we get an isomorphism

$$H^1(A, L \times S^1) \cong H^1(F, L), \tag{4.1}$$

and also

$$H^1(D^4 \setminus F, S^3 \setminus L) = 0.$$

In addition, we see from the Mayer-Vietoris sequence that the isomorphism in (4.1) is induced by the inclusion of $\partial \nu F$ into $\nu F$.

The relative Gysin sequence can also be applied to the pair $(D^4 \setminus F, S^3 \setminus L)$ with the real line bundle associated to the double cover. The relevant part of the Gysin sequence is

$$H^1(D^4 \setminus F, S^3 \setminus L) \to H^1(\widetilde{D^4 \setminus F}, \widetilde{S^3 \setminus L}) \to H^1(D^4 \setminus F, S^3 \setminus L), \tag{4.2}$$

showing that $H^1(\widetilde{D^4 \setminus F}, \widetilde{S^3 \setminus L}) = 0$.

This can be used to calculate the Betti numbers of $N$, which is constructed from the double cover of $D^4 \setminus F$ by attaching $\nu F$. Applying the Mayer-Vietoris sequence again gives

$$0 \to H^1(N, \partial N) \to H^1(\widetilde{D^4 \setminus F}, \widetilde{S^3 \setminus L}) \oplus H^1(F, L) \to H^1(A, L \times S^1) \to \dots.$$

Combining this with (4.1) we see that $H^1(N, \partial N) = 0$. Since $N$ is compact and orientable with rational homology sphere boundary, we have

$$b_1(N) = b_3(N) = 0.$$

The Euler characteristic of $N$ is given by $\chi(N) = 2\chi(D^4) - \chi(F) = 2 - n$, from which we see that $b_2(N) = 1 - n$. □

This result allows us to prove the following statement about 2-bridge links.

**Corollary 4.2.** *Let $S(p, q)$ be a two-bridge link. If $F$ is a smoothly properly embedded surface in $D^4$ with $\chi(F) = 1$ and no closed components, bounded by $S(p, q)$ then the link also bounds a ribbon embedding of $F$.*

*Proof.* Assume $p$ is even, since the odd case was established in [Lis07a]. In order to have the correct Euler characteristic and number of boundary components, $F$ must be the union of a disk and a Möbius band. By Proposition 4.1, the double cover of $D^4$ branched over $F$ is a rational homology ball and is bounded by the lens space $L(p, q)$. By a result of Lisca [Lis07a, Theorem 1.2], there is a ribbon embedding of $F$ in $D^4$. □

Restricting to the case of an oriented surface, still not necessarily connected, we can apply the same ideas used in Proposition 4.1 to $2^k$-fold cyclic branched covers.

**Corollary 4.3.** *Let $F$ be an oriented surface, locally flatly embedded in $D^4$ with no closed components and Euler characteristic $n$. Let $k > 0$ and suppose that $F$ bounds an oriented link whose Alexander polynomial is non-zero at each $2^k$-th root of unity. Then the $2^k$-fold cyclic branched cover of $D^4$ with branch set $F$ has $b_1 = b_3 = 0$ and $b_2 = 2^k - 1 - (2^k - 1)n$.*

*Proof.* We show this using the proof of Proposition 4.1 and induction on $k$. For $k = 1$, this is the proposition above. Let $(D_k, S_k)$ be the $2^k$-fold cover of $(D^4 \setminus F, S^3 \setminus L)$. There is an action of $\mathbb{Z}/2^k$ on $(D_k, S_k)$. From this, we have an action of $\mathbb{Z}/2$ and the quotient is $(D_{k-1}, S_{k-1})$. By induction we may assume that $H^1(D_{k-1}, S_{k-1}; \mathbb{Z}/2) = 0$ and so (4.2) shows that $H^1(D_k, S_k; \mathbb{Z}/2) = 0$ as well.

We may now follow the remainder of the proof of Proposition 4.1 as the assumption on the Alexander polynomial of $L$ guarantees that the $2^k$-fold cyclic branched cover of $L$ is a rational homology sphere. □

Proposition 4.1 motivates the following definition.

**Definition 4.4.** *A link $L \subset S^3$ is called* $\chi$-slice *if it is the boundary of a properly embedded surface $F$ (with no closed components) in $D^4$ with $\chi(F) = 1$.*

Some examples of $\chi$-slice links are shown in Figure 4.3. Note that for knots this just agrees with the usual definition of slice as a surface with Euler characteristic one and one boundary component must be a disk. Proposition 4.1 implies that the double branched cover of every $\chi$-slice link with non-zero determinant bounds a rational ball.

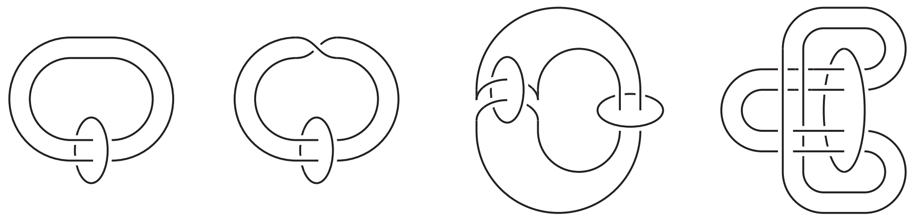


Figure 4.3: **The connected sum of two Hopf links bounds a disk and an annulus; the $(2,4)$ torus link bounds a disk and a Möbius band; the Borromean rings bound two disks and a punctured torus; the connected sum of the Hopf and Whitehead links bounds a disk and an annulus.**

Ideally we would like to quotient out by $\chi$-slice links. This turns out to be problematic, for reasons we will discuss in Section 4.3. Instead we use a more restricted notion. We will consider the cases of partly oriented links and marked oriented links separately. For the most part, we will consider smooth surfaces.

**Remark 4.5.** Some other work has considered links bounding orientable surfaces with Euler characteristic one [Ore02, Flo04, Flo05]. Baader [Baa12] has defined a notion of cobordism distance between oriented links such that $\chi$-sliceness is equivalent to cobordism distance zero from the unknot. Recently Gilmer-Livingston [GL11] and Batson [Bat12] have considered non-orientable surfaces bound by a knot.

## 4.1 Partly oriented links and smooth surfaces

**Definition 4.6.** *A pair of partly oriented links $L_0, L_1$ are called smoothly $\chi$-*concordant *if $-L_0\#L_1$ bounds a smooth surface $F$ which is properly embedded in $D^4$, has no closed components and is a union of annuli, Möbius bands and one disk whose boundary is the marked component of $-L_0\#L_1$.*

We will show that this gives an equivalence relation and leads to a link concordance group $\mathcal{L}$. First, we describe the relation in terms of embedded surfaces in $S^3 \times [0,1]$.

**Lemma 4.7.** *Partly oriented links $L_0$, $L_1$ are $\chi$-concordant if and only if there exists a smoothly properly embedded surface $F_0$ in $S^3 \times [0,1]$ satisfying*

- *$F_0$ is a disjoint union of annuli, including one oriented annulus $A$, and Möbius bands;*
- $F_0 \cap S^3 \times \{i\} = L_i \times \{i\}, \quad i = 0, 1;$
- $\partial A = \vec{K}_1 \times \{1\} \cup \vec{K}_0^r \times \{0\}$*, where $\vec{K}_i$ is the oriented component of $L_i$ and $\vec{K}_0^r$ denotes the knot $\vec{K}_0$ with the opposite orientation.*

*Proof.* This follows from Definition 4.6 as in the knot case. The pairs $(D^4, F)$ and $(S^3 \times [0,1], F_0)$ can be obtained from each other by drilling out an arc of $A$ or attaching a $(3,1)$-handle pair. □

**Lemma 4.8.** *$\chi$-concordance is an equivalence relation.*

*Proof.* For any partly oriented link $L$, $-L\#L$ is $\chi$-nullconcordant ($\chi$-concordant to the unknot) by the usual argument for knots. Taking the description in Lemma 4.7, we set $F_0 = L \times [0,1]$.

Symmetry of the relation is immediate as applying an orientation reversing diffeomorphism to the four-ball takes a surface bounded by $-L_0\#L_1$ to one bounded by $-L_1\#L_0$. Transitivity follows by composing the cobordisms $F_0$ from Lemma 4.7. After discarding any closed components, we are left with annuli and Möbius bands. □

**Lemma 4.9.** *The set of smooth $\chi$-concordance classes of partly oriented links forms an abelian group $\mathcal{L}$ under connected sum along the marked component. The knot concordance group $\mathcal{C}$ is a direct summand with complement $\mathcal{L}_0$ consisting of links with a slice marked component. The map*

$$\mathcal{L} \to \mathcal{C} \oplus \mathcal{L}_0$$

*taking a link $L$ with marked component $\vec{K}$ to*

$$\left([\vec{K}], [-\vec{K}\#L]\right)$$

*is an isomorphism.*

*Proof.* Connected sum is well-defined, commutative, and associative for partly oriented links by a variant of the usual proof for knots (see e.g. [BZ03, Chapter 7A]). Suppose that $L$, $L_0$ and $L_1$ are partly oriented links, and that $L_0 \sim L_1$. Let $F_0$ be the cobordism in $S^3 \times [0,1]$ between $L_0$ and $L_1$, as in Lemma 4.7, with oriented annulus component $A$. A copy of $L\times[0,1]$ can be embedded parallel to $A$ and so we can take the connected sum along $A$. This shows that $L_0\#L \sim L_1\#L$ and that connected sum gives a well-defined operation on $\mathcal{L}$. The class of the identity is represented by the unknot and the inverse of $[L]$ is $[-L]$. The inclusion of oriented knots into partly oriented links induces a monomorphism $\mathcal{C} \to \mathcal{L}$ since for knots $\chi$-concordance agrees with the usual definition of knot concordance. The homomorphism

$$[L] \longmapsto [\vec{K}],$$

taking the $\chi$-concordance class of a partly oriented link to the concordance class of its oriented component is a splitting map for the inclusion. It follows that the direct complement $\mathcal{L}_0$ consists of classes $[L]$ where the oriented component of $L$ is slice. For any partly oriented link $L$ with oriented component $\vec{K}$ we have $[-\vec{K}\#L] \in \mathcal{L}_0$ and

$$L \sim \vec{K}\#-\vec{K}\#L$$

by associativity, giving the claimed isomorphism. □

We can obtain an invariant of $\chi$-concordance by considering the linking numbers of components.

**Lemma 4.10.** *Let $L$ be a link in $S^3$ bounding a smoothly properly embedded surface $F$ in $D^4$, and suppose that $F = F_1 \sqcup F_2$ is a disjoint union. This gives a decomposition of $L$ into $L_1 \sqcup L_2$, where $L_i = \partial F_i$. Then the total mod 2 linking number of $L_1$ with $L_2$ is zero, i.e.*

$$\sum_{K_i \text{ in } L_i} \operatorname{lk}(K_1, K_2) \equiv 0 \pmod 2.$$

*Proof.* By isotopy of $F$, if necessary, we may assume the radial distance function $r$ on $D^4$ restricts to give a Morse function on $F$. For $i = 1, 2$, let $(F_i)_t$ be the level set of $r$ restricted to $F_i$ at height $t$. We can arrange that $(F_i)_t = \emptyset$ for $t < \frac{1}{2}$. The total linking number of each component of $(F_1)_t$ with each component of $(F_2)_t$ modulo 2 is constant with respect to $t$ as the sum does not change at regular values, maxima or minima and changes by an even number at a saddle point of $F$. Since this is zero for $t = \frac{1}{3}$, the value at $t = 1$,

$$\sum_{K_i \text{ in } L_i} \operatorname{lk}(K_1, K_2) \equiv 0 \pmod 2.$$

□

Lemma 4.10 shows that there is a homomorphism

$$l : \mathcal{L} \longrightarrow \mathbb{Z}/2$$

given by taking

$$l([L]) = \sum_{K' \neq \vec{K}} \mathrm{lk}(\vec{K}, K'),$$

where $\vec{K}$ is the oriented component of $L$. The class of the Hopf link $H$ (with one marked oriented component) has $H = -H$ and $l(H) = 1$, and generates a $\mathbb{Z}/2$ summand of $\mathcal{L}_0$.

Proposition 4.1 shows that taking double branched covers gives a group homomorphism

$$\Sigma_2 : \mathcal{N} \to \Theta^3_{\mathbb{Q}},$$

where $\mathcal{N}$ is the subgroup of $\mathcal{L}$ consisting of classes represented by links with nonzero determinant. Note in particular that the determinant of any $\chi$-slice link is a square.

We will use this fact to show that $\mathcal{L}_0$ contains a $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty$ subgroup by looking at 2-component 2-bridge links. These all represent classes in $\mathcal{L}_0$ because each component is a one-bridge knot, and so an unknot.

**Proposition 4.11.** *The two-bridge links* $\{S(q^2+1, q) \,|\, q \text{ odd}\}$ *generate a* $(\mathbb{Z}/2)^\infty$ *subgroup of* $\mathcal{L}_0$.

*Proof.* Each partly oriented link $L = S(q^2+1, q)$ for $q$ odd satisfies $L = -L$ and therefore has order at most two in $\mathcal{L}$. Since $q^2+1$ is not a square the order is two.

We will show that the subgroup of $\mathcal{L}_0$ generated by $\{S(q^2+1, q) \,|\, q \text{ odd}\}$ is infinitely generated and hence is isomorphic to $(\mathbb{Z}/2)^\infty$. Suppose it is generated by some finite subset $S = \{S(q_i^2+1, q_i)\}$. Choose a prime $p$ congruent to 1 modulo 4 which does not divide $q_i^2+1$ for any $i$. Since $-1$ is a quadratic residue modulo $p$ there exists an odd positive $q < p$ with $q^2+1$ divisible by $p$ but not by $p^2$. Then the connected sum of $S(q^2+1, q)$ with any linear combination of elements of $S$ has determinant divisible by $p$ but not $p^2$ and so this determinant is not a square. It follows using Proposition 4.1 that $S(q^2+1, q)$ is not in the subgroup of $\mathcal{L}_0$ generated by $S$ and this shows that we have an infinitely generated subgroup of $\mathcal{L}_0$. □

**Proposition 4.12.** *(Corollary of [Lis07b, Theorem 1.1]) The subgroup of the rational homology cobordism group* $\Theta^3_{\mathbb{Q}}$ *generated by lens spaces*

$$\{L(2k, 1) \,|\, k > 2\}$$

*is independent in* $\Theta^3_{\mathbb{Q}}$.

*Proof.* This follows from [Lis07b, Theorem 1.1] since for $k > 2$, $L(2k, 1)$ is not contained in any of Lisca's families $\mathcal{R}$ or $\mathcal{F}_n$. □

From Propositions 4.1 and 4.12 we see that the two-bridge links

$$\{S(2k, 1) \,|\, k > 2\},$$

generate a $\mathbb{Z}^\infty$ subgroup of $\mathcal{L}_0$.

The results of this section can be combined to give the following statement, describing the basic features of $\mathcal{L}$.

**Theorem 4.13.** *The set of smooth $\chi$-concordance classes of partly oriented links forms an abelian group*

$$\mathcal{L} \cong \mathcal{C} \oplus \mathcal{L}_0$$

*under connected sum which contains the smooth knot concordance group $\mathcal{C}$ as a direct summand. The inclusion $\mathcal{C} \hookrightarrow \mathcal{L}$ is induced by the inclusion of oriented knots into partly oriented links.*

*The complement $\mathcal{L}_0$ of $\mathcal{C}$ in $\mathcal{L}$ contains a $\mathbb{Z}/2$ direct summand and a $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty$ subgroup.*

## 4.2 Marked oriented links and smooth surfaces

For marked oriented links, we make a similar definition.

**Definition 4.14.** *A pair of marked oriented links $L_0, L_1$ are called smoothly $\chi$-*concordant *if $-L_0\#L_1$ bounds a smooth oriented surface $F$ which is properly embedded in $D^4$, has no closed components and is a union of annuli and one disk whose boundary is the marked component of $-L_0\#L_1$.*

**Remark 4.15.** This definition is largely the same as in the partly oriented case and we will use the same terminology in each case.

In this section, we will establish a result analogous to Theorem 4.13.

**Theorem 4.16.** *The set of smooth $\chi$-concordance classes of marked oriented links forms an abelian group*

$$\widetilde{\mathcal{L}} \cong \mathcal{C} \oplus \widetilde{\mathcal{L}}_0$$

*under connected sum which contains the smooth knot concordance group $\mathcal{C}$ as a direct summand (with $\mathcal{C} \hookrightarrow \widetilde{\mathcal{L}}$ induced by the inclusion of oriented knots into marked oriented links). Forgetting orientations on nonmarked components induces an epimorphism $\widetilde{\mathcal{L}} \to \mathcal{L}$. We obtain group homomorphisms as in Figure 4.4, which are induced from maps in Figure 4.2.*

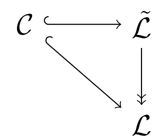


Figure 4.4: **Relationship between concordance groups.**

*The complement $\widetilde{\mathcal{L}}_0$ of $\mathcal{C}$ in $\widetilde{\mathcal{L}}$ contains a $\mathbb{Z} \oplus \mathbb{Z}/2$ direct summand and a $\mathbb{Z}^\infty$ subgroup.*

We can modify Lemma 4.7 to adapt it to this case – now $F_0$ is a collection of properly embedded annuli, including a 'marked' annulus $A$ connecting the marked components. For marked oriented links $L_0$ and $L_1$, this shows that if they are $\chi$-concordant, they must have the same number of components modulo 2.

Similarly, Lemmas 4.8 and 4.9 and their proofs can be adapted simply by replacing any reference to partly oriented links with marked oriented links. We denote the smooth concordance group of marked oriented links $\widetilde{\mathcal{L}}$. Recall that there is a map from the set of marked oriented links to the set of partly oriented links forgetting the orientation of the unmarked components. This map commutes with connected sum and it is evident from the definitions that this descends to a surjective homomorphism from $\widetilde{\mathcal{L}}$ to $\mathcal{L}$ as a collection of properly embedded annuli and a disk bound by the marked component gives a perfectly adequate $\chi$-nullconcordance for the partly oriented variant. The embedding of $\mathcal{C}$ into $\mathcal{L}$ is the composition of this map with the embedding of $\mathcal{C}$ into $\widetilde{\mathcal{L}}$.

The map $l : \mathcal{L} \to \mathbb{Z}/2$ lifts to a map on $\widetilde{\mathcal{L}}$.

**Lemma 4.17.** *Let $L$ be a $\chi$-nullconcordant marked oriented link with marked component $K$. Then $\sum \operatorname{lk}(K, K') = 0$, where the sum is taken over all components $K' \neq K$ of $L$.*

*Proof.* This follows by modifying the proof of Lemma 4.10 where $F$ is a surface in $D^4$ given by the $\chi$-nullconcordance and $F_1$ the disk component bounded by $K$. In the oriented case the sum of linking numbers between the level sets of $F_1$ and $F_2$ does not change at any critical point of $r|_F$. $\square$

This gives a homomorphism

$$\tilde{l} : \widetilde{\mathcal{L}} \to \mathbb{Z}$$

which is a lift of $l : \mathcal{L} \to \mathbb{Z}/2$ by taking the total linking number with the marked component. There is also a homomorphism $\mu$ to $\mathbb{Z}/2$ where $\mu(L)$ is the number of components of $L$ modulo 2.

We can use these maps to get a $\mathbb{Z} \oplus \mathbb{Z}/2$ summand of $\widetilde{\mathcal{L}}_0$. The marked oriented (positive) Hopf link $\tilde{H}$ in Figure 4.1 has $l = \mu = 1$ and the marked oriented two component unlink $U$ has $l = 0$, $\mu = 1$ and order two in $\widetilde{\mathcal{L}}$. These two links therefore generate such a summand.

A $\mathbb{Z}^\infty$ subgroup of $\widetilde{\mathcal{L}}_0$ is generated by marked oriented two-bridge links

$$\{S(2k, 1) \,|\, k > 2\}.$$

This follows by the same argument as for $\mathcal{L}$, after we choose an orientation of these links.

This completes the proof of Theorem 4.16.

**Remark 4.18.** The two-bridge links $S(q^2 + 1, q)$ which were shown in Proposition 4.11 to generate a subgroup $(\mathbb{Z}/2)^\infty < \mathcal{L}_0$ have infinite order in $\widetilde{\mathcal{L}}$ since they map to nonzero values under $\tilde{l}$.

The Levine-Tristram signature gives another tool for studying $\widetilde{\mathcal{L}}$. Let $\omega \in S^1 \backslash \{1\}$ be a prime-power root of unity and $M$ be a Seifert matrix for $L$. The Levine-Tristram signature $\sigma_\omega$ and nullity $n_\omega$ of $L$ are defined as the signature and nullity of $(1-\bar{\omega})M + (1-\omega)M^T$. The signature is additive under connected sum of marked oriented links and changes sign under $L \mapsto -L$. The nullity is also additive with respect to connected sum and is invariant under reversing orientation. Let $\widetilde{\mathcal{N}}_\omega$ be the subgroup consisting of elements with a representative with zero Levine-Tristram nullity $n_\omega$ – this is equivalent to saying $\omega$ is not a root of the one-variable Alexander polynomial.

**Lemma 4.19.** *Let $L$ be an oriented link with $n_\omega(L) = 0$. If $L$ is $\chi$-slice then $\sigma_\omega(L) = 0$. It follows that the Levine-Tristram signature gives a homomorphism*

$$\sigma_\omega : \widetilde{\mathcal{N}}_\omega \to \mathbb{Z}.$$

*Proof.* The vanishing of the Levine-Tristram signature for a $\chi$-slice link with $n_\omega(L) = 0$ follows directly from the Murasugi-Tristram inequality, see [Tri69, Theorem 2.27], also

[KT76,Gil93,Flo05,CF08]. Since the signature is additive with respect to connected sums, we get a homomorphism. □

Turaev showed that there is a bijection between the set of quasi-orientations (orientations up to overall reversal) on a link $L$ in $S^3$ and the set of spin structures on the double-branched cover $\Sigma_2(S^3, L)$ [Tur88, §2.2]. The following result extends this map to orientable surfaces in the four-ball. The proof is modelled on [Tur88].

**Proposition 4.20.** *Let $F$ be an oriented smoothly, properly embedded surface in $D^4$ and let $N$ be the double cover of $D^4$ branched along $F$. There is a natural bijective correspondence between the set of quasiorientations of $F$ and the set of spin structures on $N$. The spin structure on $\partial N$ determined by the induced orientation on the link $L = \partial F \subset S^3$ admits an extension over $N$, which is unique if $F$ has no closed components.*

*Proof.* Write $F = F_1 \cup \ldots \cup F_m$ where $\{F_i\}$ are the components of $F$. Let $D_i$ be an oriented meridional disk for $F_i$ and $\mu_i = \partial D_i$. These generate $H_1(D^4 \setminus F; \mathbb{Z}) \cong \mathbb{Z}^m$.

The map $\gamma : H_1(D^4 \setminus F) \to \mathbb{Z}/2$ given by sending each $\mu_i$ to 1, defines the double cover

$$\pi : N \setminus \tilde{F} \to D^4 \setminus F$$

where $\tilde{F}$ is the preimage of $F$ in $N$. A loop $l$ in $D^4 \setminus F$ lifts to a loop in $N \setminus \tilde{F}$ if and only if $\gamma([l])$ is even. Thus we can define an element of $H^1(N \setminus \tilde{F}; \mathbb{Z}/2) \cong \mathrm{Hom}(H_1(N \setminus \tilde{F}), \mathbb{Z}/2)$ by

$$h = \frac{\gamma \circ \pi_*}{2} \pmod 2. \tag{4.3}$$

Choose a framing $f_i$ for each meridional disk $D_i$ and the induced framing on $\mu_i$. Let $\tilde{\mu_i}$ and $\tilde{D_i}$ be the preimages of $\mu_i$ and $D_i$ in $N$. Note that these also have an induced framing. A spin structure $\mathfrak{s}$ on $N \setminus \tilde{F}$ extends uniquely to one on $N$ if and only if the restriction of $\mathfrak{s}$ to the framed submanifold $(\tilde{\mu_i}, \widetilde{f_i|_{\mu_i}})$ extends over $(\tilde{D_i}, \tilde{f_i})$ for each $i$.

There are two spin structures on $S^1$. The frame bundle $\mathrm{Fr}(S^1)$ is a copy of $S^1$ and these spin structures correspond to the two double covers. The spin structure on $D^2$ restricts to give the non-trivial spin structure on $S^1$. The pullback of this spin structure to the non-trivial double cover of $S^1$ is the trivial spin structure.

Let $\tilde{\mathfrak{s}}$ be the spin structure on $N \setminus \tilde{F}$ obtained by pulling back the restriction to $D^4 \setminus F$ of the unique spin structure on $D^4$. This spin structure restricts to the non-trivial spin structure on each $\mu_i$. We see that $\tilde{\mathfrak{s}}$ restricts to the trivial spin structure on $\tilde{\mu_i}$ and hence does not extend over $\tilde{D_i}$. Since $h(\tilde{\mu_i}) = 1$ for each $i$, the spin structure $\tilde{\mathfrak{s}} + h$ does extend over $N$. This gives a bijection between the sets of quasi-orientations of $F$ and spin structures on $N$. Following the argument of [Tur88], we see that if we change the orientation of a component $F_i$ while keeping the orientation of $F_j$ constant, the value of $h$ changes on a lift of $\mu_i + \mu_j$. This shows the assignment is injective and we can see that it is surjective since the number of quasi-orientations of $F$ is the same as the number of spin structures on $N$. The latter number can be calculated as it is the order of $H^1(N;\mathbb{Z}/2)$ and in this case it is $2^{m-1}$ (see for example [LW95, Theorem 1]).

Tuarev's bijection between the spin structures on the double branched cover of a link $L$ and the quasi-orientations of the link is defined in the same way. The map $\gamma_L : H_1(S^3 \setminus L) \to \mathbb{Z}/2$ takes the meridian of each component to 1 and this defines $h_L \in H^1(S^3 \setminus L;\mathbb{Z}/2)$ as in (4.3). He shows that the spin structure obtained by taking the pullback of the one on $S^3$ and twisting by $h_L$ extends uniquely over $\partial N$. When the orientation of the link is obtained from a surface $F$, it is clear that this spin structure is the restriction of $\tilde{\mathfrak{s}} + h$ for this orientation of $F$. This extension is unique when $F$ has no closed components as then the restriction map from the set of quasi-orientations of $F$ to the set of quasi-orientation of $\partial F$ is injective.

□

The double branched cover of an oriented link then defines a spin manifold. The group $\Theta^3_{\mathbb{Q},\mathrm{Spin}}$ consists of smooth spin rational homology cobordism classes of spin rational homology three-spheres under connected sum. Two spin rational homology three-spheres $Y_0$ and $Y_1$ are spin rational homology cobordant if $-Y_0 \# Y_1$ bounds a spin rational homology four-ball, or equivalently if there is a spin rational homology cobordism $W : Y_0 \to Y_1$.

For a marked oriented link $L$ with nonzero determinant, Turaev's map (as described in the proof of Proposition 4.20) gives a spin structure $\mathfrak{s}_L$ on $\Sigma_2(S^3, L)$. Following the definition, the spin structure we get on $\Sigma_2(S^3, L\#L')$ is the same as the one obtained from $(\Sigma_2(S^3, L), \mathfrak{s}_L)\#(\Sigma_2(S^3, L'), \mathfrak{s}_{L'})$. If the marked oriented links $L$ and $L'$ are $\chi$-concordant, Propositions 4.1 and 4.20 show that $(\Sigma_2(S^3, L), \mathfrak{s}_L)$ and $(\Sigma_2(S^3, L'), \mathfrak{s}_{L'})$ are spin rational homology cobordant. The map taking $L$ to the spin manifold $(\Sigma_2(S^3, L), \mathfrak{s}_L)$ therefore gives a group homomorphism

$$\widetilde{\Sigma_2} : \widetilde{\mathcal{N}} \to \Theta^3_{\mathbb{Q},\mathrm{Spin}}$$

from the subgroup of $\widetilde{\mathcal{L}}$ represented by links with nonzero determinant to the spin rational homology cobordism group of spin rational homology three-spheres.

**Remark 4.21.** The double branched cover of a knot always has a unique spin structure, as it is a $\mathbb{Z}/2$ homology sphere. The distinction between $\Theta^3_{\mathbb{Q}}$ and $\Theta^3_{\mathbb{Q},\mathrm{Spin}}$ is thus irrelevant in the knot case.

We define an invariant of a marked oriented link $L$ with nonzero determinant by

$$\delta([L]) = 4\, d \circ \widetilde{\Sigma_2}([L]) = 4d(\Sigma_2(S^3, L), \mathfrak{s}_L),$$

where $d$ is the correction term invariant from Heegaard-Floer homology [OzSz03a].

This is a homomorphism from $\widetilde{\mathcal{N}}$ to $\mathbb{Q}$, since both $\widetilde{\Sigma_2}$ and $d$ are homomorphisms. When $L$ is a knot, this is double the concordance invariant introduced in [MO07]. We can see from [KT76] and Proposition 4.20 that $(\Sigma_2(S^3, L), \mathfrak{s}_L)$ is the boundary of the spin four-manifold given as the double branched cover of $D^4$ along a Seifert surface for $L$ and that the signature of this manifold is equal to the signature of $L$. By [OzSz03a, Theorem 1.2], it follows that $\delta(L)$ is an integer and is congruent to minus the signature of $L$ modulo 8.

For alternating links, a stronger statement can be made.

**Lemma 4.22.** *Let $L$ be a nonsplit oriented quasi-alternating link. Then $\sigma(L) + \delta(L) = 0$.*

This is proved in [DO12, Lemma 3.4] for alternating links, and is generalised to quasi-alternating links in [LO13].

The homomorphisms $\tilde{l}$, $\sigma$ and $\delta$ can be used to find a summand of $\widetilde{\mathcal{N}_0}$ – the subgroup of $\widetilde{\mathcal{L}}$ represented by links with nonzero determinant and slice marked component.

The marked oriented links $\tilde{H}$ and $\tilde{L}_1$ from Figure 4.1 have

$$(\tilde{l}, \sigma, \delta)(\tilde{H}) = (1, -1, 1)$$
$$(\tilde{l}, \sigma, \delta)(\tilde{L}_1) = (1, 0, 0).$$

Let $M$ be the Montesinos link given by plumbing twisted bands according to the positive definite plumbing graph shown in Figure 4.5. This is a three-component link with determinant four and every component is an unknot. The values of $\sigma$ and $\delta$ for the four quasiorientations on $M$ may be computed using the plumbing graph ( [Sav00, Theorem 5], [OzSz03b, Corollary 1.5]) and are

$$\sigma = -8, 0, 0, 4$$
$$\delta = 0, 0, 0, -4.$$

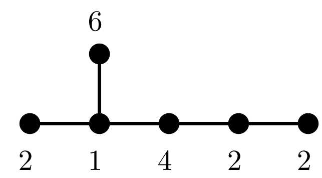



Figure 4.5: **Plumbing diagram for $M$.**

It follows that $\tilde{H}$, $\tilde{L}_1$ and $M$ (with some choice of orientation) generate a $\mathbb{Z}^3$ summand of the direct complement $\widetilde{\mathcal{N}}_0$ of $\mathcal{C}$ in $\widetilde{\mathcal{N}}$.

Corollary 4.3 allows us to define homomorphisms on subgroups of $\widetilde{\mathcal{N}}$. Let $\widetilde{\mathcal{N}}_n$ be the set of classes in $\widetilde{\mathcal{L}}$ with a representative whose Alexander polynomial is non-zero on each $2^n$-th root of unity so that $\widetilde{\mathcal{N}}_1 = \widetilde{\mathcal{N}}$. On these subgroups, there is a homomorphism

$$\Sigma_{2^n} : \widetilde{\mathcal{N}}_n \to \Theta^3_{\mathbb{Q}}.$$

## 4.3 Difference between $\chi$-slice and $\chi$-concordance

An obvious first attempt to generalise $\mathcal{C}$ would be to take the quotient by $\chi$-slice links in the monoid of either partly oriented or marked oriented links. In order to ensure that connected sum on the marked components is well-defined on the quotient we need to add the condition that the marked component should bound a disk (see Lemma 4.9). However, this relation turns out to be uninteresting.

**Proposition 4.23.** *Let $L$ be a marked oriented link with marked component $\vec{K}$ and let $H$ be the Hopf link. Then there is an unlink $U$ and $l \in \mathbb{Z}$ such that $L\# - \vec{K}\#lH\#U$ bounds a properly embedded surface $F$ with $\chi(F) = 1$ and with the marked component bounding a disk.*

*Proof.* The oriented component $K'$ of $L\# - \vec{K}$ is slice and so it bounds a disk. We choose $l$ so that the linking number of $K'$ with the rest of the link $L' = L\# - \vec{K}\#lH$ is zero. Then we can find an oriented surface bound by $L' \setminus K'$ which has only transverse intersections with the disk bounded by $K'$ and algebraic intersection number zero.

We can remove these intersection points in pairs by adding handles, giving an orientable surface bounded by $L'$. By adding new handles if needed, we can assume that the Euler characteristic is negative. We then add as many disjoint disk components to the surface as are needed to increase the Euler characteristic to one. This has the effect on the boundary of changing $L'$ to $L'\#U$ for an unlink $U$. □

Every odd component unlink bounds a union of a disk and a collection of annuli. If we were to take the quotient by $\chi$-slice links, all that we need to consider is the concordance class of the marked component, the total linking number of the marked component with the rest of the link and the parity of the number of components.

Proposition 4.23 also holds when $L$ is a partly oriented link, although we may take $l \in \{0, 1\}$ by allowing a non-orientable surface. In this case, $H$ has order two and every unlink is $\chi$-slice so the class of a link is determined by the marked component and the linking number of the marked component with the rest of the link modulo 2.

## 4.4 Topological versions

In [DO12], topological versions of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are defined, by allowing locally flat surfaces as well as smooth ones. This gives topological link concordance groups $\mathcal{L}_{\mathrm{TOP}}$ and $\widetilde{\mathcal{L}}_{\mathrm{TOP}}$. They have similar properties to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ and we get topological versions of Theorems 4.13 and 4.16.

**Theorem 4.24.** *The set of locally flat $\chi$-concordance classes of partly oriented links forms an abelian group*

$$\mathcal{L}_{\mathrm{TOP}} \cong \mathcal{C}_{\mathrm{TOP}} \oplus (\mathcal{L}_{\mathrm{TOP}})_0$$

*under connected sum which contains the topological knot concordance group $\mathcal{C}_{\mathrm{TOP}}$ as a direct summand (with $\mathcal{C}_{\mathrm{TOP}} \hookrightarrow \mathcal{L}_{\mathrm{TOP}}$ induced by the inclusion of oriented knots into partly oriented links).*

*The complement $(\mathcal{L}_{\mathrm{TOP}})_0$ of $\mathcal{C}_{\mathrm{TOP}}$ in $\mathcal{L}_{\mathrm{TOP}}$ contains a $(\mathbb{Z}/2)^\infty$ subgroup.*

**Theorem 4.25.** *The set of locally flat $\chi$-concordance classes of marked oriented links forms an abelian group*

$$\widetilde{\mathcal{L}}_{\mathrm{TOP}} \cong \mathcal{C}_{\mathrm{TOP}} \oplus (\widetilde{\mathcal{L}}_{\mathrm{TOP}})_0$$

*under connected sum which contains the topological knot concordance group $\mathcal{C}_{\mathrm{TOP}}$ as a direct summand (with $\mathcal{C}_{\mathrm{TOP}} \hookrightarrow \widetilde{\mathcal{L}}_{\mathrm{TOP}}$ induced by the inclusion of oriented knots into marked oriented links). Forgetting orientations on nonmarked components induces a surjection $\widetilde{\mathcal{L}}_{\mathrm{TOP}} \to \mathcal{L}_{\mathrm{TOP}}$.*

*The complement $(\widetilde{\mathcal{L}}_{\mathrm{TOP}})_0$ of $\mathcal{C}_{\mathrm{TOP}}$ in $\widetilde{\mathcal{L}}_{\mathrm{TOP}}$ contains a $\mathbb{Z}/2$ direct summand and a $\mathbb{Z}^\infty$ subgroup.*

*Proof of Theorem 4.24.* The proof is largely the same as in the smooth case, in particular Lemmas 4.8 and 4.9 apply without modification. Proposition 4.1 gives us a topological version of the branched double cover homomorphism $\Sigma_2$. Proposition 4.11 shows that the two-bridge links $\{S(q^2+1,q)\}$ generate a $(\mathbb{Z}/2)^\infty$ subgroup in $(\mathcal{L}_{\mathrm{TOP}})_0$. □

*Proof of Theorem 4.25.* This follows the proof of Theorem 4.16 but, as the results of [Lis07b] do not apply, we use Levine-Tristram signatures to establish that the two-bridge links

$$\{S(2k,1)\,|\,k>0\},$$

oriented so that the linking number is $+k$, are linearly independent in $\widetilde{\mathcal{L}}_{\mathrm{TOP}}$.

The Levine-Tristram signatures of these links are computed by Przytycki in [Prz11, Example 11]. In particular, it is shown that $\sigma_\omega(S(2k,1))$ is a locally constant function of $\omega$ and changes when $\psi=(1-\omega)/|1-\omega|$ satisfies $\psi^{4k}=1$ and $\psi\neq\pm1$. Suppose $\sum_{i=1}^n a_i\sigma(S(2k_i,1)) = 0$ for some integers $a_i$, with $0<k_1<\cdots<k_n$ and $a_n\neq 0$. Choosing $\omega$ such that $\psi=\exp(it)$ with $t\in[\pi/2k_n,\pi/2k_{n-1}]$, we find

$$\sum_{i=1}^n a_i\sigma_\omega(S(2k_i,1)) = a_n(\sigma_\omega(S(2k_n,1))-\sigma(S(2k_n,1)))\neq 0.$$

Linear independence in $\widetilde{\mathcal{L}}_{\mathrm{TOP}}$ then follows from Lemma 4.19, which also holds in the locally flat case. □

The smooth and topological knot concordance groups are distinct. It is also interesting to compare the smooth and topological versions of these link concordance groups.

**Theorem 4.26.** *Let $K$ be an alternating knot with negative signature (for example the right handed trefoil), and let $C$ be a knot with Alexander polynomial one and $\delta(C)\neq 0$ (such as the Whitehead double of the right handed trefoil [MO07, Theorem 1.5]). The partly-oriented links $L_2\#H$ and $L_3\#H$ shown in Figure 4.6 are trivial in $\mathcal{L}_{\mathrm{TOP}}$ and nontrivial in $\mathcal{L}$.*

*Orienting all components of $L_2\#H$ and $L_3\#H$ results in marked oriented links which are trivial in $\widetilde{\mathcal{L}}_{\mathrm{TOP}}$ and nontrivial in $\widetilde{\mathcal{L}}$, under the same hypotheses on $K$ and $C$.*

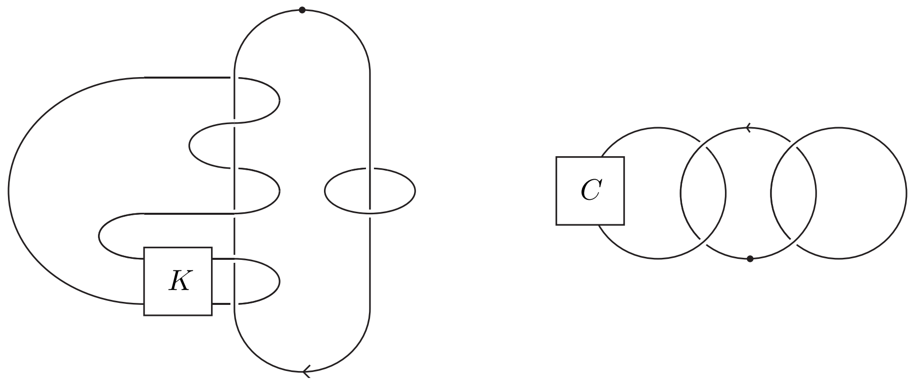



Figure 4.6: **Partly oriented links $L_2\#H$ and $L_3\#H$. The band shown passing through the box marked $K$ is tied in the knot $K$ with zero framing (cf. [CKRS10]).**

*Proof.* Each of the links shown in Figure 4.6 is a connected sum of a two-component partly oriented link $L_i$ and the Hopf link $H$, where the linking number between the components of each $L_i$ is $\pm 1$.

Suppose that the partly oriented link $L_i\#H$ is smoothly $\chi$-nullconcordant. Then it bounds a smoothly embedded surface $F$ in $D^4$ which is either one disk and two Möbius bands or a disk and an annulus. In either case with the marked component bounds the disk. The first possibility is ruled out by linking numbers as in Lemma 4.10, and the second is equivalent to existence of a concordance in the traditional sense, given by two properly embedded annuli in $S^3 \times I$, between $L_i$ and $H$. This is ruled out in the case of $L_3$ since $\delta(C) \neq 0$ implies that $C$ is not slice, and is ruled out in the case of $L_2$ by recent work of Cha-Kim-Ruberman-Strle [CKRS10].

Both $L_2$ and $L_3$ have Alexander polynomial one ( [CKRS10]) and are therefore locally flatly concordant (in the traditional sense and hence also $\chi$-concordant) to the Hopf link by a theorem of Davis [Dav06]. It follows that $L_i\#H$ is trivial in $\mathcal{L}_{\mathrm{TOP}}$ and (with an appropriate choice of orientation) in $\widetilde{\mathcal{L}}_{\mathrm{TOP}}$. $\square$

# Chapter 5

# Heegaard-Floer correction terms of lens spaces

## 5.1 $d$ invariants of lens spaces

As we discussed in the introduction, Lisca examined lens spaces which bound smooth rational balls and discovered a condition which fully characterised such lens spaces. We consider Heegaard-Floer $d$ invariants of lens spaces in the hope that they can be compared to a condition of Casson-Gordon [CG86] for a lens space to bound a rational ball.

Ozsváth and Szabó [OzSz03a] give a reciprocity formula for the $d$ invariants of lens spaces. With respect to an identification of $Spin^c(L(p,q))$ with $\mathbb{Z}/p = \{0, 1, \ldots, p-1\}$ which arises naturally from a Heegaard triple they show that for $p > q > 0$ and $0 \leq i < p+q$

$$d(-L(p,q),i) + d(-L(q,p),i) = \frac{(2i+1-p-q)^2 - pq}{4pq}. \tag{5.1}$$

**Remark 5.1.** Our orientation convention for lens spaces differs from the one used in [OzSz03a] and [JRX13]. As a result, we state some formulae for $-L(p,q)$.

This formula is similar to reciprocity formulae in number theory for Dedekind sums. This is used in [JRX13] to give a formula for the $d$ invariants of lens spaces in terms of Dedekind and Dedekind-Radamacher sums. For a real number $x$ let $\overline{B}_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$. Here $\lfloor x \rfloor$ is the floor function so $\overline{B}_1(x)$ is just the representative in $[-\frac{1}{2}, \frac{1}{2})$ for the class of $x - \frac{1}{2}$ in $\mathbb{R}/\mathbb{Z}$.

For coprime integers $p, q$ and any integer $n$, the Dedekind-Radamacher sum[1] $s(q,p;n)$

[1] This is a version of the Dedekind-Radamacher sum; it is sometimes defined using the sawtooth function

is defined by

$$s(q,p;n) = \sum_{k=0}^{|p|-1} \overline{B}_1\left(\frac{kq+n}{p}\right)\overline{B}_1\left(\frac{k}{p}\right).$$

This sum is unchanged by replacing $q$ and $n$ with $q+lp$ and $n+mp$ for integers $l,m$. The classical Dedekind sum $s(q,p)$ is given by $s(q,p;0)-\frac{1}{4}$. Jabuka-Robins-Xinli use reciprocity formulae to express the $d$ invariant in terms of these sums.

**Theorem 5.2** ( [JRX13, Theorem 1.2])**.** *Let $p,q$ be coprime positive integers and $n \in \mathbb{Z}$. Then*

$$d(-L(p,q),n) = 2s(q,p;n) + s(q,p) - \frac{1}{2p}. \tag{5.2}$$

This result is obtained by showing that the two expressions obey the same reciprocity formula. The formula (5.1) is also used in [JRX13] to find conditions on $i,j$ such that $d(L(p,q),i) \pm d(L(p,q),j)$ is zero. The following is a slight generalisation of [JRX13, Theorem 1.4].

**Theorem 5.3.** *Let $p,q$ be coprime integers and $i,j \in \mathbb{Z}/p$.*

*If $d(-L(p,q),i) - d(-L(p,q),j) \in \frac{1}{2}\mathbb{Z}$ then $p|2(i-j)(i+j-q+1)$.*

*If $d(-L(p,q),i) + d(-L(p,q),j) \in \frac{1}{2}\mathbb{Z}$ then $p|((i-j)^2+(i+j-q+1)^2)$.*

The statement differs from [JRX13, Theorem 1.4] only by changing $\{0\}$ to $\frac{1}{2}\mathbb{Z}$. For convenience, we briefly summarise the proof of one statement.

*Proof.* Suppose that $d(-L(p,q),i) - d(-L(p,q),j) \in \frac{1}{2}\mathbb{Z}$. By (5.1),

$$\begin{aligned}
&2pq\left[d(-L(p,q),i) + d(-L(q,p),i) - d(-L(p,q),j) - d(-L(q,p),j)\right] \\
&= \frac{(2i+1-p-q)^2-pq}{2} - \frac{(2j+1-p-q)^2-pq}{2} \\
&= \frac{4i^2+41-4ip-4iq-4j^2-4j+4jp+4jq}{2} \\
&= 2\left[i^2-j^2+(i-j)(1-p-q)\right] \\
&= 2(i-j)(i+j+1-p-q).
\end{aligned}$$

Since $d(-L(q,p),i)$ is always an integer multiple of $\frac{1}{2q}$ – this follows from the definition of the $d$ invariant in [OzSz03a] and is also shown in [JRX13, Lemma 2.2] using (5.2) – our assumption implies that both sides of this equation are integers divisible by $p$. We then see that $p|2(i-j)(i+j-q+1)$.

The second statement is proved similarly. □

instead of $\overline{B}_1$.

### 5.1.1 Lens spaces with square order

From now on, let $p = m^2$. This case is especially interesting as it includes all lens spaces which bound smooth rational 4-balls.

**Proposition 5.4.** *Suppose $2i+1-q = lm$ for some integer $l$. Then $d(-L(m^2, q), i) \in \frac{1}{4}\mathbb{Z}$. Further, if $m$ is odd, it is integer-valued while if $m$ is even and $l$ is odd, it is half an integer.*

*Proof.* Suppose $2i + 1 - q = lm$. By (5.1),

$$\begin{aligned} 2m^2q\left[d(-L(m^2, q), i) + d(-L(q, m^2), i)\right] &= \frac{(2i + 1 - q - m^2)^2 - m^2q}{2} \\ &= \frac{(lm - m^2)^2 - m^2q}{2} \\ &= \frac{m^2(l^2 + m^2 - 2lm - q)}{2}. \end{aligned}$$

Since $d(-L(p, q), i)$ is always a rational number in $\frac{1}{2p}\mathbb{Z}$, both sides of this equation are integers. Suppose that either $m$ or $l$ is odd. The right hand side is an integer multiple of $m^2$. Reducing modulo $m^2$, the left side is $2qm^2d(-L(m^2, q), i)$. Thus $2qm^2d(-L(m^2, q), i)$ is divisible by $m^2$. If $m$ is odd, $2q$ is coprime to $m^2$ and so $d(-L(m^2, q), i)$ is an integer while if $m$ is even $d(-L(m^2, q), i)$ is half an integer.

If $m$ and $l$ are even we see that $2qm^2d(-L(m^2, q), i)$ is divisible by $\frac{m^2}{2}$. Arguing as above, since $d(-L(m^2, q), i)$ is an integer multiple of $\frac{1}{2m^2}$ and $q$ is odd, we see that $d(-L(m^2, q), i) \in \frac{1}{4}\mathbb{Z}$. □

In particular, note that for odd $m$ $L(m^2, q)$ has $m$ integer-valued $d$ invariants while for even $m$ there are $2m$ with quarter integer values.

On the other hand, we may adapt [JRX13, Corollary 1.8] to get an upper bound.

**Corollary 5.5.** *Let $m, q$ be coprime integers. Then $L(m^2, q)$ has at most $m$ half integer-valued $d$ invariants and at most $m$ with values in $\frac{1}{4}\mathbb{Z}\backslash\frac{1}{2}\mathbb{Z}$.*

*Proof.* Theorem 5.3 gives conditions on $i, j$ if $d(-L(m^2, q), i) \pm d(-L(m^2, q), j)$ is half an integer.

The two conditions together imply, as argued in [JRX13, Corollary 1.8], that $i \equiv j$ modulo $m$.

□

Summarising, we have the following statement.

**Proposition 5.6.** *If $m$ is odd, $L(m^2, q)$ has exactly $m$ integer-valued $d$ invariants. If $m$ is even, there are exactly $m$ half-integer $d$ invariants and $2m$ quarter-integer $d$ invariants.*

We shall later see that all of the half-integer $d$ invariants are actually even integers. In the meantime, we can note the following corollary.

**Corollary 5.7.** *Suppose $d(L(m^2, q), i) \in \frac{1}{2}\mathbb{Z}\backslash\{0\}$. Then $L(m^2, q)$ does not smoothly bound a rational ball.*

*Proof.* $L(m^2, q)$ bounds a rational ball only if $m$ of its $d$ invariants vanish. Thus all $m$ of the half-integer valued $d$ invariants are zero. □

### 5.1.2 Lens spaces with square order $m^2$ and $m$ vanishing $d$ invariants

We examine the Dedekind-Radamacher sum formula for the $d$ invariants. Note that the formula in Theorem 5.2 only depends on $n$ through $s(q, p; n)$.

**Theorem 5.8.** *The $d$ invariants $d(-L(m^2, q), n + lm)$ are equal for each $l \in \mathbb{Z}$ if and only if*

$$\left\{ \left\lfloor \frac{n + iq}{m} \right\rfloor \ \textit{mod}\ m \right\}_{i=0}^{m-1} = \{i\}_{i=0}^{m-1}. \tag{5.3}$$

*Proof.* By Theorem 5.2 we have $s(q, m^2; n) = s(q, m^2; n + lm)$. Let $\{x\}$ denote the fractional part of a rational number $x$. Note that

$$\begin{aligned} s(q, p; n) &= \sum_{k=0}^{m^2-1} \left[ \left( \left\{ \frac{kq + n}{m^2} \right\} - \frac{1}{2} \right) \left( \left\{ \frac{k}{m^2} \right\} - \frac{1}{2} \right) \right] \\ &= \sum_{k=0}^{m^2-1} \left[ \left\{ \frac{kq + n}{m^2} \right\} \left\{ \frac{k}{m^2} \right\} - \frac{1}{2} \left( \left\{ \frac{kq + n}{m^2} \right\} + \left\{ \frac{k}{m^2} \right\} \right) + \frac{1}{4} \right] \end{aligned}$$

As $k$ runs between 0 and $m^2 - 1$ both

$$\left\{ \frac{kq + n}{m^2} \right\} \text{ and } \left\{ \frac{k}{m^2} \right\}$$

take every value in

$$\left\{ 0, \ldots, \frac{m^2 - 1}{m^2} \right\}.$$

Therefore $s(q, m^2, n) = s(q, m^2, n')$ whenever

$$\sum_{k=0}^{m^2-1} \left\{ \frac{kq + n}{m^2} \right\} \left\{ \frac{k}{m^2} \right\} = \sum_{k=0}^{m^2-1} \left\{ \frac{kq + n'}{m^2} \right\} \left\{ \frac{k}{m^2} \right\}. \tag{5.4}$$

Let $n' = n + jqm$ for some $j \in \mathbb{Z}$. Choose $\frac{r}{m^2}$ for some $0 \leq r < m^2$. This residue appears in the sums in equation (5.4) as

$$\left\{\frac{kq+n}{m^2}\right\} \text{ and as } \left\{\frac{k_j q + n + jqm}{m^2}\right\},$$

for some $k, k_j$. Modulo $m^2$ we have

$$kq \equiv k_j q + jqm,$$

and so $k \equiv k_j + jm$.

The coefficients of $\frac{r}{m^2}$ in the two sums in equation (5.4) are multiples of $\frac{1}{m^2}$ between zero and $\frac{m^2-1}{m^2}$. If $n$ is changed to $n' = n + jqm$, the coefficient of $\frac{r}{m^2}$ is reduced by $\frac{j}{m}$ if

$$\frac{r}{m^2} = \left\{\frac{kq+n}{m^2}\right\}$$

for some $k \geq m(m-j)$. Otherwise the coefficient increases by $1 - \frac{j}{m}$.

If $s(q, p; n) = s(q, p; n + lm)$ for every $l \in \mathbb{Z}$, it follows that

$$\sum_{k=0}^{m-1} \left\{\frac{kq+n}{m^2}\right\} = \sum_{k=m}^{2m-1} \left\{\frac{kq+n}{m^2}\right\} = \ldots = \sum_{k=m^2-m}^{m^2-1} \left\{\frac{kq+n}{m^2}\right\}. \tag{5.5}$$

Since $kq + n$ is the same as $(k+m)q + n$ modulo $m$, the set

$$\left\{\left\{\frac{kq+n}{m^2}\right\}\right\}$$

contains one value in

$$\left\{0, \frac{1}{m^2}, \ldots \frac{m-1}{m^2}\right\}.$$

By the same reasoning, the same is true for each set

$$\left\{\frac{i}{m}, \frac{im+1}{m^2}, \ldots \frac{im+m-1}{m^2}\right\},$$

with $0 \leq i < m$, from which (5.3) follows.

Conversely, (5.3) immediately implies (5.5) as $m, q$ are coprime. Any difference between $s(q, p; n)$ and $s(q, p; n + lm)$ arises from permuting these sums of residues so we see that these Dedekind-Radamacher sums must be equal and thus that the $d$ invariants are also equal. $\square$

The correction terms of lens spaces can also be computed from a plumbing diagram [OzSz03b]. J. Greene explained to me how to use this approach to obtain the following result. Indeed, a similar method can also be used to show that Lisca's diagonalisation condition (for lens spaces) translates to one on the $d$ invariants. The $d$ invariants of the manifold are the same as the $d$ invariants of a certain lattice, which for lens spaces is the same as the standard definite lattice. A detailed description appears in [Gre11].

**Proposition 5.9.** *Suppose*

$$d(L(m^2, q), n) = d(L(m^2, q), n + lm)$$

*for each* $l \in \mathbb{Z}$. *Then all of these* $d$ *invariants are zero.*

*Proof.* By (5.3), for $0 \leq i \leq m-1$, we can write $\{n + iq\}$ as $\{ma_i + b_i\}$ where $\{b_i\} = \{0, \ldots, m-1\}$ and $\{a_i \bmod m\} = \{0, \ldots, m-1\}$. The sum is

$$\sum_{i=0}^{m-1} n + iq = nm + \frac{1}{2}qm(m-1).$$

This is equal to

$$m\sum_{i=0}^{m-1} a_i + \sum_{i=0}^{m-1} b_i = m\left(\frac{1}{2}m(m-1) + km\right) + \frac{1}{2}m(m-1),$$

for some $k \in \mathbb{Z}$. We then have

$$2n - q + 1 = m(m - q + 2k).$$

Modulo $m$, $2n - q + 1$ is zero. When $m$ is even $q$ is odd, so $2n - q + 1$ is an odd multiple of $m$. For either parity of $m$ this condition specifies $n$ mod $m$. By Proposition 5.4 this implies that $d(L(m^2, q), n + lm)$ is half an integer.

Let $\Lambda$ be a definite lattice whose discriminant group is $\mathbb{Z}/m^2$. There is a unimodular integral lattice $Z$ such that $\Lambda \subset Z \subset \Lambda^*$. This is given by the preimage of the unique $\mathbb{Z}/m$ subgroup of $\Lambda^*/\Lambda$. We know it is integral because there is only one linking form on a cyclic group and it vanishes on this subgroup. The $d$ invariants corresponding to $Z/\Lambda$ can be computed using characteristic vectors on $Z$. Since it is unimodular, the square of any characteristic vector is the same as the rank modulo 8 and these $d$ invariants are even integers.

The $d$ invariants of $L(m^2, q)$ are calculated by a definite lattice $\Lambda_{m^2,q}$. If there is a set of $m$ equal $d$ invariants, they are precisely the $d$ invariants which have half-integer values. It follows from Proposition 5.6 that these must be the invariants calculated by the unimodular lattice $Z$. Up to replacing $L(m^2, q)$ by $L(m^2, m^2 - q)$ we can assume that each $d(L(m^2, q), n + lm) \geq 0$. By a theorem of Elkies [Elk95] the $d$ invariant of a unimodular lattice is at most 0 and so these $d$ invariants are zero. □

### 5.1.3 Casson-Gordon condition

Casson and Gordon [CG86, Corollary on p.188] give a necessary condition for $L(m^2, q)$ to bound a rational ball when $m$ is odd. This condition is observed to also be sufficient for $m \leq 105$. We can translate the condition to one which appears similar to (5.3).

To state their condition, we require the following definitions. Let $x, y, \in \mathbb{R}$ and $\Delta(x, y)$ be the triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(x, 0)$ and $(x, y)$. A count $\mathrm{Int}\Delta(x, y)$ of the integer points in the triangle is given by counting integer points in the interior with multiplicity one; integer points on the edges with multiplicity $\frac{1}{2}$; integer points at vertices with multiplicity $\frac{1}{4}$ and the point $(0, 0)$ with multiplicity zero.

**Proposition 5.10** ( [CG86]). *Suppose $S(p, q)$ is a ribbon knot. Then $p = m^2$ and for each $r = 1, \ldots, m-1$*

$$\mathit{Area}\left(\Delta\left(mr, \frac{qr}{m}\right)\right) - \mathit{Int}\left(\Delta\left(mr, \frac{qr}{m}\right)\right) = \pm\frac{1}{4}. \tag{5.6}$$

By [Lis07a], $S(p, q)$ is ribbon if and only $L(p, q)$ smoothly bounds a rational ball, so we can restate the hypothesis in the terms of the lens space. We can reinterpret the condition as follows.

**Corollary 5.11.** *Suppose $L(m^2, q)$ smoothly bounds a rational ball, with $m$ odd. For $r = 1, \ldots, m-1$ let*

$$n_r = \left\lceil \frac{qr}{m} \right\rceil = \frac{qr}{m} + \frac{s_r}{m}.$$

*Then there is an $\epsilon_r \in \{0, 1\}$ such that*

$$\left|\left\{ i \in [0, mr] \cap \mathbb{Z} \,|\, \left\{\frac{iq}{m^2}\right\} > \left\{\frac{qr}{m}\right\}\right\}\right| = rs_r - \epsilon_r. \tag{5.7}$$

*Proof.* The square $S$ in Figure 5.1 has edges parallel to the axes and vertices at $(0, 0)$ and $(mr, n_r)$ and it decomposes into two copies of $\Delta\left(mr, \frac{qr}{m}\right)$ and a parallelogram $A$. Counting the vertex $(mr, n_r)$ with multiplicity zero, we can easily see that

$$\mathrm{Int}\left(\Delta\left(mr, \frac{qr}{m}\right)\right) = \frac{1}{2}\left(\mathrm{Int}(S) - \mathrm{Int}(A)\right).$$

By identifying the left and right edges of $S$ and the top and bottom edges, we see that $\mathrm{Int}(S) = mrn - \frac{1}{2}$, where each integer point contributes one apart from the point obtained from identifying all the vertices of $S$, which contributes two quarters and two zeros. Since the gradient of the diagonal line of $\Delta$ has slope $\frac{q}{m^2}$ and length at most $m^2 - m$, there are

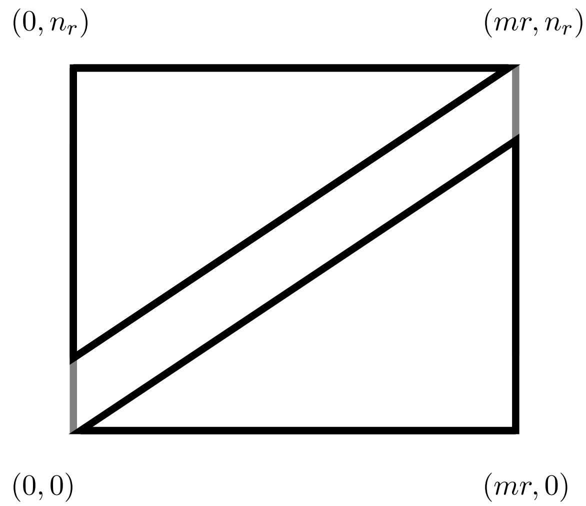



Figure 5.1: **The square $S$ decomposes as two copies of $\Delta\left(mr, \frac{qr}{m}\right)$ and a parallelogram $A$.**

never any integer points on this line. Consequently, all the integer points of $A$ are in the interior. Above an integer $i$ on the $x$-axis, there is an integer point in $A$ if and only if the interval

$$\left[\frac{iq}{m^2}, \frac{iq}{m^2} + \frac{s_r}{m}\right]$$

contains an integer. Thus the left side of (5.7) is

$$\left|\left\{i \in [0, mr] \cap \mathbb{Z} \mid \left\{\frac{iq}{m^2}\right\} > \left\{\frac{qr}{m}\right\}\right\}\right| = \mathrm{Int}(A).$$

The area of $\Delta\left(mr, \frac{qr}{m}\right)$ is $\frac{1}{2}qr^2$ so using (5.6) in the second step

$$\begin{aligned}\mathrm{Int}(A) &= \mathrm{Int}(S) - 2\mathrm{Int}\left(\Delta\left(mr, \frac{qr}{m}\right)\right) \\ &= mrn_r - \frac{1}{2} - qr^2 \pm \frac{1}{2} \\ &= mr\left(\frac{qr}{m} + \frac{s_r}{m}\right) - qr^2 - \epsilon_r \\ &= rs_r - \epsilon_r.\end{aligned}$$

□

We may interpret condition (5.7) in terms of the function taking $i \in \{0, \ldots m^2 - 1\}$ to the fractional part of $\frac{iq}{m^2}$. The condition says that the values here are distributed in a way which can be imprecisely described as 'fairly even'. We sample the first $mr$ values and compare to a threshold $1 - \frac{s_r}{m}$. Of our $m^2$ values of $i$, there are $s_r m$ with corresponding fractional part above this threshold and our condition is that roughly as many as we would expect lie in the first $mr$.

This is similar to condition (5.3), which can also be interpreted as saying that the values of

$$i \mapsto \left\{ \frac{iq}{m^2} \right\}$$

are distributed evenly. This condition is that, if we start from $n$, the values of this function on a set of $m$ consecutive numbers lie in different intervals $\left[\frac{j}{m}, \frac{j+1}{m}\right)$.

It would be interesting to compare these conditions for odd $m$. For $m < 105$, Casson and Gordon showed that their condition is equivalent to $L(m^2, q)$ smoothly bounding a rational ball but it is unknown if this holds for all $m$.

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