DEFINITE MANIFOLDS BOUNDED BY RATIONAL HOMOLOGY THREE SPHERES

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Abstract. This paper is based on a talk given at the McMaster University Geometry and Topology conference, May 2004. The basic question we address is whether a given definite form may be the intersection pairing of a smooth four-manifold bounded by a given rational homology sphere. We survey various obstructions including linking pairings, rho invariants and the more recent Floer homology invariants of Frøyshov and Ozsváth and Szabó. We describe various examples and compute the four ball genus of some families of knots and links.

Introduction. We first review some basic facts about rational homology spheres and four-manifolds that they bound. In the next section we describe applications of non-bounding results to questions regarding knots and links. In the remaining sections we discuss obstructions to bounding that arise from linking pairings, rho invariants and gauge theory. All manifolds in this paper are assumed to be smooth and oriented.

Recall that every three-manifold is the boundary of some four-manifold; a simply connected such four-manifold arises from any (integral) surgery description of the three-manifold. A rational homology three sphere Y is a closed three-manifold with $b_1(Y) = 0$. Suppose Y is the boundary of a four-manifold X. The intersection pairing Q_X on $H_2(X;\mathbb{Z})/\text{Tors}$ is a nondegenerate symmetric bilinear integer-valued form. We say that the form Q_X is bounded by Y. Note that every nondegenerate $\mathbb{Z}$-valued form is bounded by some rational homology sphere. For a given Y we wish to study the set of forms Q which Y bounds. Recall that a form Q on a free abelian group A is positive (resp. negative) definite if $Q(x,x) > 0$ (resp. $Q(x,x) < 0$) for all nonzero $x \in A$. We will be particularly interested in constraints on definite forms bounded by Y.

Examples. • Donaldson's theorem [1] tells us that the only definite forms that S^3 bounds are the diagonal unimodular forms.
• Any lens space $L(p,q)$ bounds both positive and negative definite forms.
• Any Seifert fibred rational homology sphere bounds at least one definite form.

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These examples suggest that one may hope to find an obstruction to Y bounding a definite form of one sign; or if Y bounds definite forms of both signs one can hope to find a restriction on the type of a definite form of one sign. We exhibit some examples of these kinds in the last section.

Applications to knots and links. Let L be an oriented link with μ components in the three sphere; denote its signature by $\sigma(L)$. A slice surface F for L is a connected surface properly embedded in B^4 with boundary $\partial F = L$. The four-ball genus $g^*(L)$ is defined to be the minimal genus of any slice surface. This is clearly bounded above by the Seifert genus of L.

Let X be the double branched cover of B^4 along a minimal genus slice surface F. Then the boundary of X is the double cover of S^3 branched along L; denote this by Y. Suppose that Y is a rational homology sphere (for example this will be the case if L is a knot). It is shown by Kauffman and Taylor in [9] that $b_2(X) = 2g(F) + \mu - 1$, and the signature of X is equal to $\sigma(L)$. Thus we have

$$g^*(L) \geq \frac{|\sigma(L)| - \mu + 1}{2}. \tag{1}$$

(This bound is originally due to Murasugi [11].) If equality holds then X has a definite intersection pairing of rank $|\sigma(L)|$. We also note that X is a spin manifold (see [12]). Thus if there is an obstruction to Y bounding an even definite form of rank $|\sigma(L)|$, we must have strict inequality in (1).

Potentially stronger bounds on $g^*(L)$ may be obtained by replacing $|\sigma(L)|$ in (1) by $|\sigma_\omega(L)| + n_\omega(L)$; here $\omega \in S^1 - \{1\}$, $\sigma_\omega(L)$ is the Tristram-Levine signature and $n_\omega(L)$ is the nullity. The resulting bound is called the Murasugi-Tristram inequality (see e. g. [5]). In the case that the Alexander polynomial has no roots on $S^1 - \{1\}$, then these bounds are equal to that in (1).

The four-ball genus of a knot is a lower bound for the unknotting number $u(K)$, which is the minimum number of crossing changes in any diagram of K which are necessary to yield the unknot. Bounds on unknotting number may also be attained using the Montesinos trick, which describes the effect of a crossing change on the double branched cover of a knot. In particular if a knot has $u(K) = 1$ then the double cover Y of S^3 branched along K is obtained by $\pm(2m-1)/2$ Dehn surgery along a knot K^*, where $m > 0$ and $2m - 1$ is the determinant of K. Hence Y bounds the definite rank two form represented by the matrix

$$Q = \pm \begin{pmatrix} m & 1 \\ 1 & 2 \end{pmatrix}.$$

Ozsváth and Szabó use this as part of an obstruction to $u(K) = 1$ in [17].

Bilinear forms and lattices. Let $Q : A \times A \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a free abelian group A. The form Q is said to be *even* if $Q(x, x) \in 2\mathbb{Z}$

for all $x \in A$. The pair (A, Q) can be thought of as an integral lattice L in the space $V = A \otimes \mathbb{R}$ with a pairing $x \cdot y$ induced by Q. Let $L^* = \{x \in V \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}$ denote the dual lattice; clearly $L \subset L^*$.

A *characteristic covector* for Q is an element ξ of the dual lattice for which $\xi \cdot y \equiv y \cdot y \pmod 2$ for all $y \in L$. A *characteristic vector* is a characteristic covector which is an element of L. The set of characteristic covectors is $\{\xi + 2x \mid x \in L^*\}$, where ξ is a characteristic covector. If Q is even one may take $\xi = 0$.

Linking pairings on finite groups. Let H be a finite abelian group. A nondegenerate symmetric bilinear form $\lambda : H \times H \to \mathbb{Q}/\mathbb{Z}$ is called a *linking pairing* on H. We say that a bilinear form $L = (A, Q)$ *presents* (H, λ) if there is a short exact sequence

$$L \longrightarrow L^* \longrightarrow H,$$

and $\lambda(x, y) = -\tilde{x} \cdot \tilde{y} \pmod 1$ where $\tilde{x}, \tilde{y} \in L^*$ map to x, y in H.

Proposition 1. *Let $L_i = (A_i, Q_i)$ for $i = 1, 2$ be presentations of a linking pairing (H, λ) with A_i of rank n_i. Then $L = (A_1, Q_1) \oplus (A_2, -Q_2)$ may be embedded in a unimodular lattice (U, Q) of rank $n_1 + n_2$.*

If ξ_1 and ξ_2 are characteristic covectors for Q_1, Q_2 which both map to the same element of H, then for some $x_1 \in L_1^*$ with $2x_1 \in L_1$, $(\xi_1 + 2x_1, \xi_2)$ is a characteristic vector for Q.

Proof. Let $\{h_j\}$ be a generating set for H and let $x_{i,j} \in L_i^*$ be elements which map to h_j. Then $u_j = (x_{1,j}, x_{2,j}) \in L^*$ pair integrally with each other (and with all the elements of L). Thus $U = L + \sum_j \mathbb{Z} u_j$ is an integral lattice; this is called a *gluing* of the lattices L_1 and $-L_2$. To see that U is unimodular note that the index of L in U is $|H|$. Note also that if $z_i \in L_i^*$ then (z_1, z_2) belongs to $U^* = U$ if and only if z_1 and z_2 map to the same element of H.

Now let χ be any characteristic vector in U. Write $\chi = (\chi_1, \chi_2)$ where χ_i is a characteristic covector for Q_i. Then $\chi_i = \xi_i + 2z_i$ for some $z_i \in L_i^*$. Choose $z_1' \in L_1^*$ which maps to the same element of H as z_2 and let $x_1 = z_1 - z_1'$. Then $2x_1 \in L_1$ and $\chi - 2(z_1', z_2) = (\xi_1 + 2x_1, \xi_2)$ is a characteristic covector for Q. □

Given a linking pairing (H, λ) we define its *rho invariant* as follows. Let $L = (A, Q)$ be any presentation of (H, λ). (Wall [20] showed that presentations always exist.) Any choice of a characteristic vector $c \in L$ for Q gives rise to a map $\hat{\rho}_c : H \to \mathbb{Q}/2\mathbb{Z}$: for $x \in H$ let $\tilde{x} \in L^*$ be any lift of x and define

$$\hat{\rho}_c(x) := \frac{(c + 2\tilde{x})^2 - \sigma(Q)}{4} \pmod 2.$$

Proposition 2. *Let $L_i = (A_i, Q_i)$ be two presentations of (H, λ) and let $c_i \in L_i$ be characteristic. Then $\hat{\rho}_{c_2} = \hat{\rho}_{c_1} \circ \tau$, where $\tau : H \to H$ is translation by an element of order 2.*

Definition 3. *We call the collection of all $\hat{\rho}_c$ the* rho invariant *of (H,λ).*

Proof. Consider the unimodular lattice (U,Q) obtained from $L_1 \oplus -L_2$ as in Proposition 1; note that the signature $\sigma(Q)$ equals $\sigma(Q_1)-\sigma(Q_2)$. There is an element $x_1 \in L_1^*$ with $2x_1 \in L_1$ such that $c=(c_1+2x_1,c_2)$ is a characteristic vector for the unimodular lattice (U,Q). Given $x \in H$ choose any $\tilde{x}_i \in L_i^*$ mapping to x. Then $(\tilde{x}_1,\tilde{x}_2) \in U$, hence $c_x = c+2(\tilde{x}_1,\tilde{x}_2)$ is also characteristic; its square is

$$c_x^2 = (c_1+2x_1+2\tilde{x}_1)^2 - (c_2+2\tilde{x}_2)^2.$$

Since Q is unimodular it follows that $c_x^2 - \sigma(Q) \equiv 0 \pmod 8$ (see e.g. [18]). Substituting for c_x^2 and $\sigma(Q)$ shows that $\hat{\rho}_{c_1}(x+t) = \hat{\rho}_{c_2}(x)$, where $t \in H$ is the image of x_1. □

A complete algebraic classification of linking pairings was obtained by Wall [20] and Kawauchi-Kojima [7]. Another general property relevant to our application is the following stable equivalence result.

Theorem 4 ([2, 8, 21])**.** *Two lattices $L_i=(A_i,Q_i)$ present the same linking pairing if and only if there exist unimodular lattices U_i with $L_1 \oplus U_1 \cong L_2 \oplus U_2$.*

Linking pairings and rho invariants of 3-manifolds. The *linking pairing* of a rational homology sphere Y is a nondegenerate symmetric bilinear pairing

$$\lambda : H^2(Y;\mathbb{Z}) \times H^2(Y;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

induced by Poincaré duality. Let X be any four-manifold bounded by Y. Then for any x,y in the image of the restriction map $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$,

$$\lambda(x,y) = -\tilde{x}\cdot\tilde{y} \pmod 1,$$

where $\tilde{x},\tilde{y} \in H^2(X;\mathbb{Z})$ are any classes whose restriction to Y are x,y, and $\cdot$ denotes the intersection pairing. In particular, if the restriction map is surjective (e.g., if X is simply connected), the intersection pairing on $H^2(X;\mathbb{Z})$ gives a presentation for the linking pairing of Y.

Let the intersection pairing on $H_2(X;\mathbb{Z})/\mathrm{Tors}$ be represented in a chosen basis by a matrix Q. The natural map $H^2(X,Y) \to H^2(X)$ induces a map from $H^2(X,Y)/\mathrm{Tors}$ to $H^2(X)/\mathrm{Tors}$ which with respect to the corresponding bases is given by Q. Thus the quotient of $H^2(X)$ by the sum of its torsion subgroup and the image of $H^2(X,Y)$ is isomorphic to $\mathbb{Z}^n/Q(\mathbb{Z}^n)$, where $n=b_2(X)$. Let H denote $H^2(Y)$ and let $h=|H|$ and $\delta = |\det Q|$. Then $h=\delta t^2$, where t is the order of the image T of the torsion subgroup of $H^2(X)$ in $H^2(Y)$ (see [12] for more details). The restriction map from $H^2(X)$ to H induces a monomorphism

$$\psi : \mathbb{Z}^n/Q(\mathbb{Z}^n) \longrightarrow H/T. \qquad (2)$$

It follows that on the subgroup $\psi(\mathbb{Z}^n/Q(\mathbb{Z}^n)) + T$ of H the linking pairing λ is T-invariant and is determined by Q. *Hence:* given a rational homology sphere Y and a form Q one may find an obstruction to Y bounding a four-manifold with intersection pairing Q by checking whether there exist such a subgroup T of H and a homomorphism ψ.

Finally we note that it follows from Theorem 4 that if Q presents the linking pairing of Y then Y stably bounds Q. More precisely, we have the following

Corollary 5. *Let Q be a symmetric bilinear form that presents the linking pairing of a rational homology sphere Y. Then Y bounds a four-manifold X whose intersection pairing is $Q \oplus U$, where U is some unimodular pairing.*

Proof. Let X_1 be any simply connected manifold that Y bounds. Then its intersection pairing Q_1 presents λ_Y, hence by Theorem 4 there exist unimodular pairings U_1 and U such that $Q_1 \oplus U_1 = Q \oplus U$. We may assume U_1 to be indefinite odd, thus $U_1 = n\langle 1\rangle \oplus m\langle -1\rangle$. Then $X = X_1 \# n\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$ has the required form of the intersection pairing. □

To define the rho invariant of Y we need to discuss spin^c structures. Recall that there is a free and transitive action of $H^2(Y;\mathbb{Z})$ on $\mathrm{Spin}^c(Y)$. Thus if we fix a basepoint $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ the action gives an identification $H^2(Y;\mathbb{Z}) \cong \mathrm{Spin}^c(Y)$ which takes $x \in H^2(Y;\mathbb{Z})$ to the tensor product of $\mathfrak{s}$ with the line bundle x; we denote this spin^c structure by $\mathfrak{s}+x$. Note that the Chern class of $\mathfrak{s}+x$ is $c_1(\mathfrak{s})+2x$. Similarly a choice of basepoint $\mathfrak{t} \in \mathrm{Spin}^c(X)$ gives an identification $\mathrm{Spin}^c(X) \cong H^2(X;\mathbb{Z})$.

The *rho invariant* of Y is a map

$$\rho : \mathrm{Spin}^c(Y) \to \mathbb{Q}/2\mathbb{Z}$$

defined as follows. Let X be a four-manifold bounded by Y for which the restriction map $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is onto. For any spin^c structure $\mathfrak{s}$ on Y choose a lift $\tilde{\mathfrak{s}} \in \mathrm{Spin}^c(X)$ and define

$$\rho(\mathfrak{s}) := \frac{c_1(\tilde{\mathfrak{s}})^2 - \sigma(X)}{4} \pmod 2. \tag{3}$$

That this is well defined follows as in the proof of Proposition 2; moreover, if X is any manifold with boundary Y and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is the restriction of $\tilde{\mathfrak{s}} \in \mathrm{Spin}^c(X)$, then (3) holds. In the current setting there is no ambiguity in the definition since if X_1, X_2 are two four-manifolds with $\partial X_1 = Y = \partial X_2$ and $\tilde{\mathfrak{s}}_i \in \mathrm{Spin}^c(X_i)$ are spin^c structures whose restrictions to Y agree, then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ can be glued to give a spin^c structure on the closed manifold $X_1 \cup_Y -X_2$.

Consider now any four-manifold X with boundary Y. Recall that the restriction map from $\mathrm{Spin}^c(X)$ to $\mathrm{Spin}^c(Y)$ is equivariant under conjugation of spin^c structures and the fixed points of this action are exactly the spin structures. Fix a basepoint $\tilde{\mathfrak{s}}_0 \in \mathrm{Spin}^c(X)$ whose Chern class belongs to the sum of the torsion subgroup of

$H^2(X)$ and the image of $H^2(X,Y)$. Use a spin structure $\mathfrak{s}_0$ as the basepoint in $\mathrm{Spin}^c(Y)$. With these choices the restriction map on spin^c structures induces an affine monomorphism

$$\psi' : \mathbb{Z}^n/Q(\mathbb{Z}^n) \longrightarrow H/T; \tag{4}$$

this is equal to the composition of ψ with a translation of H/T by an element of order 2. *Hence:* if Y bounds a form Q then there is a subgroup T of H and a map ψ' as above so that on the preimage in $\mathrm{Spin}^c(Y)$ of the image of ψ' (for some $\mathfrak{s}_0$), ρ is T-invariant and is determined by Q.

The linking pairing and the rho invariant of Y both constrain which forms (not just definite) may be bounded by Y. There is a close relationship between the two invariants.

Proposition 6 (c.f. Taylor [19])**.** *The linking pairing λ of Y is completely determined by the rho invariant ρ of Y. The rho invariant is determined by the linking pairing and a choice of spin structure on Y. More precisely, let $\hat\rho$ be the rho invariant of λ (see Definition 3). Then $\rho = \hat\rho_c$ for some c.*

Proof. Let X be a simply connected manifold with boundary Y; then the restriction map from $H^2(X)$ to $H^2(Y)$ is surjective. Let $\tilde{\mathfrak{s}}, \tilde{x}, \tilde{y}$ be lifts of $\mathfrak{s}, x, y$ to X. It is straightforward to check that

$$\lambda(x,y) = \frac{-\rho(\mathfrak{s}+x+y)+\rho(\mathfrak{s}+x)+\rho(\mathfrak{s}+y)-\rho(\mathfrak{s})}{2} \pmod 1.$$

Now choose as the base point in $\mathrm{Spin}^c(X)$ a spin^c structure $\tilde{\mathfrak{s}}_0 \in \mathrm{Spin}^c(X)$ whose Chern class belongs to the image of $H^2(X,Y)$. Then the restriction of $\tilde{\mathfrak{s}}_0$ to Y is a spin structure $\mathfrak{s}_0$. Compute $\hat\rho_c$ using the intersection pairing Q_X and $c = c_1(\tilde{\mathfrak{s}}_0)$. Then clearly ρ is the composition of $\hat\rho_c$ with the identification of $\mathrm{Spin}^c(Y)$ with $H^2(Y;\mathbb{Z})$ which takes $\mathfrak{s}_0$ to 0. □

Let us consider the rho invariant as an obstruction to Y bounding a form Q. By the above proposition we see that the rho invariant is equivalent to the linking pairing and a choice of spin structure. However since the identification of $\mathrm{Spin}^c(Y)$ with $H^2(Y)$ may take any spin structure to 0, we still need to test all possible homomorphisms ψ' as above. Thus the extra information contained in the rho invariant is lost when we apply it to a quadratic form and the constraint on forms bounded by Y given by the rho invariant is equivalent to that given by the linking pairing.

Constraints from Donaldson's theorem. Let Q_1, Q_2 be $\mathbb{Z}$-valued forms on free abelian groups A_1, A_2 of ranks n_1, n_2 respectively. If a rational homology sphere Y is known to bound Q_1 we may ask can it also bound Q_2. Since gluing two four-manifolds along their common boundary yields a closed four-manifold $X = X_1 \cup_Y -X_2$ with a unimodular intersection pairing, we see that if Y bounds Q_2 then $(A_1, Q_1) \oplus$

$(A_2, -Q_2)$ may be embedded in a unimodular lattice of rank $n_1 + n_2$. In the case that Q_1 is positive-definite and Q_2 is negative-definite we get a refinement of this basic obstruction from Donaldson's theorem. The closed four-manifold X is positive-definite in this case, and so its intersection form is the standard pairing on $\mathbb{Z}^{n_1+n_2}$ as a sublattice of the Euclidean space $\mathbb{R}^{n_1+n_2}$. Thus if Y bounds Q_2 then $(A_1, Q_1) \oplus (A_2, -Q_2)$ may be embedded as a sublattice of $\mathbb{Z}^{n_1+n_2}$.

In particular if (A_1, Q_1) cannot be embedded in $\mathbb{Z}^m$ for any m then Y does not bound any negative-definite form. This approach has been used by Lisca, see for example [10]. See also [13] for further discussion.

Example. Let Y be the Poincaré homology sphere, oriented as the boundary of the positive $E8$ disk bundle plumbing. Then since the $E8$ lattice cannot be embedded in $\mathbb{Z}^m$ for any m, it follows that Y does not bound any negative-definite four-manifold.

Constraints from Floer homology. Using gauge theory one can define rational-valued invariants of spin^c structures on a rational homology sphere Y that give rise to obstructions to Y bounding a definite four-manifold X. A result of this type was first proved by Frøyshov [4] in Seiberg-Witten theory. We recall the version of Ozsváth and Szabó in Heegaard Floer homology [15]: if X is positive-definite then for every spin^c structure $\mathfrak{t}$ on X,

$$c_1(\mathfrak{t})^2 - rk(H^2(X;\mathbb{Z})) \geq 4d(Y, \mathfrak{t}|_Y); \tag{5}$$

here $d(Y, \mathfrak{s})$ is called the *correction term* invariant of Y in spin^c structure $\mathfrak{s}$. Moreover, since the reduction modulo 2 of the correction term is the rho invariant of Y, the two sides of the inequality are congruent modulo 8.

Note that to apply these conditions to the question of whether a positive definite form Q may be the intersection pairing of a four-manifold X bounded by Y one needs to allow for all possible restriction maps $\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y)$ (this amounts to considering maps ψ' as in (4), see [12] for details). This is particularly simple if Y is an integer homology sphere. Then Y has a unique spin^c structure (which is a spin structure) and we denote the unique correction term for Y by $d(Y)$. Then (5) becomes

$$\xi^2 \geq n + 4d(Y),$$

where n is the rank of the form Q and ξ is any characteristic vector for Q. If $d(Y) \geq 0$ this inequality combines well with the following result.

Theorem 7 (Elkies [3])**.** *If Q is a unimodular positive-definite form of rank n, then there exists a characteristic vector satisfying $\xi^2 \leq n$. Moreover, there is a characteristic vector with square strictly less than n unless $Q = n\langle 1\rangle$.*

In particular, if $d(Y) = 0$, then the two inequalities imply $Q = n\langle 1\rangle$, which gives Donaldson's Theorem for $Y = S^3$. If $d(Y) > 0$ then Y cannot bound any positive-definite manifold; for example, this is the case for the Poincaré homology sphere oriented as the boundary of the negative $E8$ plumbing.

Short characteristic covectors. As noted above the inequality (5) is not so easy to use for rational homology spheres. Suppose that Y bounds a positive definite X. For each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ one needs to determine the smallest square of any characteristic covector corresponding to a spin^c structure on X whose restriction to Y is $\mathfrak{s}$. A weaker but more practical test is provided by the following generalisation of Theorem 7 which gives an upper bound on the length of the shortest characteristic covector of a positive definite form. Then to check for obstruction to Y bounding a definite manifold one only needs to know the correction terms of Y.

Theorem 8 ([14])**.** *Let Q be an integral positive-definite quadratic form of rank n and determinant δ. Then there exists a characteristic covector ξ for Q with*

$$\xi^2 \leq \begin{cases} n-1+1/\delta & \text{if } \delta \text{ is odd,} \\ n-1 & \text{if } \delta \text{ is even;} \end{cases}$$

moreover, there exists a characteristic covector for which the inequality is strict unless $Q = (n-1)\langle 1\rangle \oplus \langle \delta\rangle$.

Corollary 9. *Let Y be a rational homology sphere with $|H_1(Y;\mathbb{Z})| = \delta$. If Y bounds a positive-definite four-manifold X with no torsion in $H_1(X;\mathbb{Z})$ then*

$$\min_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} 4d(Y,\mathfrak{s}) \leq \begin{cases} -1+1/\delta & \text{if } \delta \text{ is odd,} \\ -1 & \text{if } \delta \text{ is even.} \end{cases}$$

The inequality is strict unless the intersection form of X is $(n-1)\langle 1\rangle \oplus \langle \delta\rangle$.

The proof of Theorem 8 is based on gluing of lattices. Using the classification of linking pairings [20, 7] we show in [14] that the sum of four copies of any lattice $L = (\mathbb{Z}^n, Q)$ embeds in a unimodular positive definite lattice U of rank $4n$. If U is not the diagonal lattice $\mathbb{Z}^{4n}$, then Theorem 7 implies the result of Theorem 8. If $U = \mathbb{Z}^{4n}$ we analyze the sublattices of index δ^2 in U (for which the exponent of the quotient group is δ) to arrive at the conclusion of Theorem 8.

Examples.

Example 10. *For any $m \geq 1$ the two-bridge link $L_m = S(16m-4, 8m-1)$, oriented as in Figure 1, has four-ball genus one.*

Proof. First note that making one crossing change yields the Hopf link which bounds a smooth annulus in the four-ball. It follows that

$$0 \leq g^*(L_m) \leq 1.$$

(One may also obtain this bound by noting that the Seifert genus of L_m is one.) The signature $\sigma(L_m) = 1$ for $m \geq 1$, and the double branched cover of L_m is the lens space $Y_m = L(16m-4, 8m-1)$. Thus if $g^*(L_m) = 0$, it follows that Y_m bounds a positive-definite even four-manifold X with $b_2(X) = 1$. We will show using the linking pairing that this is not the case.

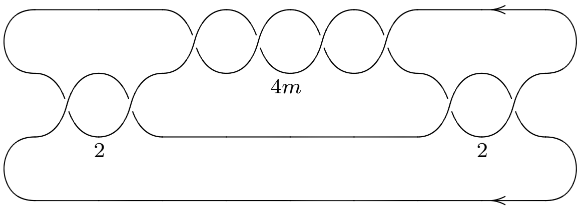

FIGURE 1. **The two-bridge link** $S(16m-4, 8m-1)$**.** The numbers indicate the number of crossings ($m = 1$ in this diagram), and the arrows specify the chosen orientation.

To compute the linking pairing λ, note that $\frac{16m-4}{8m-1}$ has a continued fraction expansion

$$\frac{16m-4}{8m-1} = 2 - \frac{1}{4m - \frac{1}{2}},$$

and so Y_m bounds a plumbing of disk bundles over S^2 according to the linear graph with weights $-2, -4m, -2$. The linking pairing on $H = H^2(Y_m)$ is presented by the intersection pairing of this plumbing, which is represented on $A = \mathbb{Z}^3$ by the matrix

$$Q = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -4m & 1 \\ 0 & 1 & -2 \end{pmatrix},$$

with inverse

$$Q^{-1} = \frac{-1}{16m-4} \begin{pmatrix} 8m-1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 8m-1 \end{pmatrix}.$$

Thus we have a short exact sequence

$$\mathbb{Z}^3 \xrightarrow{Q} \mathbb{Z}^3 \longrightarrow H.$$

The second $\mathbb{Z}^3$ is the dual lattice, on which the induced pairing is given by $x \cdot y = x^T Q^{-1} y$. A generator h of $H \cong \mathbb{Z}^3/Q(\mathbb{Z}^3)$ is the image of $(0,0,1)$ and its square is

$$\lambda(h,h) = \frac{8m-1}{16m-4}.$$

Now suppose that Y_m bounds a positive-definite even four-manifold X with $b_2(X) = 1$. The intersection pairing on X is $[s]$ for some even s. From $|H| = 4(4m-1) = st^2$ we see that $s = 4k$ for some $k \equiv 3 \pmod 4$. The equality of linking pairings yields

$$\frac{(8m-1)t^2}{16m-4} \equiv -\frac{i^2}{s} \pmod 1$$

for some i. Multiplying by s and reducing modulo k results in $i^2 \equiv -1 \pmod k$ which is a contradiction.

It is not difficult to show that the Alexander polynomial of L_m has no roots on $S^1 - 1$, thus the Murasugi-Tristram inequality gives no information for L_m. However the fact that $g^*(L_m) > 0$ may also be deduced using the nonvanishing of the Arf invariant. $\square$

For details on the following examples see [13].

Example 11. *If*

$$\frac{1}{\alpha_2}, \frac{1}{\alpha_3} < \frac{\beta_1}{\alpha_1} < \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$$

and $\alpha_3 \geq 3$, *then the Seifert fibred space* $Y = Y(-2; (\alpha_1, \beta_1), (\alpha_2, \alpha_2 - 1), (\alpha_3, \alpha_3 - 1))$ *cannot bound a negative-definite four-manifold.*

Proof. Y is the boundary of a positive-definite plumbing whose intersection pairing does not admit any embedding into $\mathbb{Z}^n$. The result now follows from Donaldson's Theorem. $\square$

Example 12. *The Montesinos knot* $K_{q,r} = M(2; (qr-1, q), (r+1, r), (r+1, r))$ *with odd* $q \geq 3$ *and even* $r \geq 2$, *has signature* $\sigma = 1 - q$ *and has*

$$g = g^* = \frac{q+1}{2}.$$

Proof. The knot $K_{q,r}$ is equal to $M(0; (qr-1, q), (r+1, -1), (r+1, -1))$ and has a spanning surface with genus $\frac{q+1}{2}$. The double branched cover Y of $K_{q,r}$ satisfies the conditions of Example 11 thus Y does not bound a negative-definite four-manifold; the genus formula follows. $\square$

Example 13. *Let* $Y_a = Y(-1; (2,1), (3,1), (a,1))$ *with* $a \geq 7$. *Then* $\delta = |H^2(Y)| = a - 6$ *and*

$$\min_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} 4d(Y, \mathfrak{s}) = \begin{cases} -1 + 1/\delta & \text{if } \delta \text{ is odd,} \\ \\ -1 & \text{if } \delta \text{ is even.} \end{cases}$$

If Y_a *bounds a positive-definite four-manifold* X *with no torsion in* $H_1(X; \mathbb{Z})$ *then the intersection pairing of* X *is equivalent to* $(n-1)\langle 1 \rangle \oplus \langle \delta \rangle$.

Proof. Y_a is the boundary of a positive-definite plumbing whose intersection pairing is equivalent to $3\langle 1 \rangle \oplus \langle \delta \rangle$. From this the formula for $d(Y, \mathfrak{s})$ follows. Then Corollary 9 implies the claim. $\square$

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